On a connection between the order of a finite group and the set of conjugacy classes size

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In this paper, all groups are finite. The number of elements of a set π is denoted by $|\pi|$. Denote the set of prime divisors of positive integer n by $\pi(n)$, and the set $\pi(|G|)$ for a group G by $\pi(G)$. The greatest power of a prime p dividing the natural number n will be denoted by n_p. For a set of prime π and a natural number n we will denote $n_\pi = \prod_{p\in\pi} |n|_p$.

Let G be a group and take $a \in G$. We denote by a^G the conjugacy class of G containing a. Put $N(G) = \{|x^G|, x \in G\} \setminus \{1\}$. Denote by the $|G||_p$ number p^n such that $N(G)$ contains α multiple of p^n and avoids the multiple of p^{n+1}. For $\pi \subseteq \pi(G)$ put $|G||_\pi = \prod_{p\in\pi} |G||_p$. For brevity, write $|G||$ to mean $|G||_{\pi(G)}$. Observe that $|G||_p$ divides $|G|_p$ for each $p \in \pi(G)$. However, $|G||_p$ can be less than $|G|_p$.

Definition. *Let p and q be distinct numbers. Say that a group G satisfies the condition $\{p,q\}^*$ and write $G \in \{p,q\}^*$ if we have $\alpha_{\{p,q\}} \in \{|G||_p, |G||_q, |G||_{\{p,q\}}\}$ for every $\alpha \in N(G)$.*

A. R Camina (see [1]) proved that a group G with $\{p,q\}^*$-property is nilpotent if $N(G) = \{1, p^n, q^m, p^n q^m\}$. A. Beltram and M.J. Filipe (see [2])extended Camina's theorem in the following way: let G be a finite soluble group whose conjugacy class sizes are $\{1, n, m, nm\}$, where n and m are coprime positive integers; then G is nilpotent and the integers n and m are prime-power numbers. Q. Kong and X. Guo (see [3]) investigated groups such that the set of conjugacy class sizes of biprimary elements is precisely $1, p^\alpha, m, p^\alpha m$, where p^α is a prime power, $(p,m) = 1$ and there is a p-element whose conjugacy class size is p^α. They proved that in this case such groups is nilpotent and $m = q^\beta$ for some prime number $q \neq p$.

In the general case, a group with the $\{p,q\}^*$-property is not nilpotent. For example, let $G \simeq L_n(k)$. Then $G \in \{p,q\}$, where p is a primitive prime divisor of $k^n - 1$ and q is a primitive prime divisor of $k^{n-1} - 1$.

In this paper we inspect the groups with $\{p,q\}^*$-properties and trivial center.

Theorem. *If $G \in \{p,q\}^*$ is a group with trivial center, where $p, q \in \pi(G)$ and $p > q > 5$, then $|G|_{\{p,q\}} = |G||_{\{p,q\}}$.*

Corollary. *In the hypotheses of the theorem, $C_G(g) \cap C_G(h) = 1$ for every p-element g and every q-element h.*

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References

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