Second eigenmatrices of a non-commutative association schemes obtained from steiner systems

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This is joint work with Akihiro Munemasa

Definition 1. Let X be a finite set of order n and $R_0, R_1, \dots, R_d \subset X \times X$. The *adjacency matrix* A_i is a $(0,1)$-matrix indexed by X such that $(A_i)_{x,y} = 1$ if $(x,y) \in R_i$ and $(A_i)_{x,y} = 0$ otherwise. Let I_n, J_n be the identity matrix and all-ones matrix of order n, respectively. The pair $(X, \{R_i\}_{i=0}^d)$ is called an *association scheme of class* d if the following hold:

(i) $A_0 = I_n$.

(ii) $\sum_{i=0}^d A_i = J_n$.

(iii) For any $i \in \{0, 1, \dots, d\}$, there exists $i' \in \{0, 1, \dots, d\}$ such that $A_{i'} = A_i^T$.

(iv) For any $i, j, k \in \{0, 1, \dots, d\}$, there exists $p_{i,j}^k$ such that $A_i A_j = \sum_{k=0^d} p_{i,j}^k A_k$.

An association scheme is commutative if $A_i A_j = A_j A_i$ holds for any $i, j \in \{0, 1, \dots, d\}$ and non-commutative otherwise. The algebra spanned by $A_0, A_1, \dots, A_d$ over $\mathbb{C}$ is called a *adjacency algebra*. Since the adjacency algebra is semisimple, it is isomorphic to $\bigoplus_{k=1}^r M_{d_k}(\mathbb{C})$ for uniquely determined positive integers $r, d_1, d_2, \dots, d_r$, where $\sum_{k=0}^r d_k^2 = n$. We may construct a set $\{E_k^{(i,j)} \mid 1 \le k \le r, 1 \le i, j \le d_k\}$ as a basis (see [1]). Since the adjacency algebra has $\{A_0, A_1, \dots, A_d\}$ as a basis, there exist $q_k^{(i,j)}(l)$ such that

$$E_k^{(i,j)} = \frac{1}{n} \sum_{l=0}^d q_k^{(i,j)}(l) A_l.$$

The square matrix $Q = (q_k^{(i,j)}(l))$ is called a second eigenmatrix.

Let V be a finite set of order v and $\mathcal{B}$ be a set of k-subsets of V. The pair $(V, \mathcal{B})$ is called a t-(v,k,λ)design if $\lambda = \#\{B \in \mathcal{B} \mid S \subset B\}$ holds for any t-subset S of V. Let $\mathcal{F} = \{(p, V) \in V \times \mathcal{B} \mid p \in B\}$. In particular, t-$(v,k,1)$-design is called a steiner system. In [2], for steiner systems with $t = 2$, $(\mathcal{F}, \{R_i\}_{i=0}^6)$ is a non-commutative association scheme of class 6.

A quotient association scheme of class 2 is constructed from the non-commutative association scheme $(\mathcal{F}, \{R_i\}_{i=0}^6)$. The quotient association scheme is commutative, and the primitive idempotents of the quotient association scheme correspond to central idempotents of $(\mathcal{F}, \{R_i\}_{i=0}^6)$. In my talk, we reveal the correspondence and construct the second eigenmatrix of $(\mathcal{F}, \{R_i\}_{i=0}^6)$.

References

[1] H. Kharaghani, S. Suda, Non-commutative association schemes and their fusion association schemes. arXiv:1706.06281.

[2] M. Klin, A. Munemasa, M. Muzychuk, P. H. Zieschang, Directed strongly regular graphs obtained from coherent algebras. *Linear Alg. and its App.* **377** (2004) 83–109.