On the semigroup of similarities with unique one point intersection, not satisfying WSP

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Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of contraction similarities in $\mathbb{R}^n$. The system $\mathcal{S}$ is said to satisfy the *open set condition* (OSC), iff there exists an open set O such that $S_i(O) \subset O$ and $S_i(O) \cap S_j(O) = \varnothing$ for all distinct $i, j \in I = \{1, \dots, m\}$.

Denote by $F = \{S_{\mathbf{i}} : \mathbf{i} \in I^\infty\}$ the semigroup, generated by $\mathcal{S}$; then $\mathcal{F} = F^{-1} \circ F$, or a set of all compositions $S_{\mathbf{j}}^{-1} S_{\mathbf{i}}$, $\mathbf{i}, \mathbf{j} \in I^*$, is the *associated family of similarities*. The system $\mathcal{S}$ has the *weak separation property* (WSP) iff $\mathrm{Id} \notin \overline{\mathcal{F} \setminus \mathrm{Id}}$. If the system doesn't have WSP, then it doesn't satisfy OSC, but the opposite is not true.

A nonempty compact set $K = K(\mathcal{S})$ such that $K = \bigcup\limits_{i=1}^{m} S_i(K)$, is called an *attractor* of the system $\mathcal{S}$, or a *self-similar set*. A set $C(\mathcal{S}) = \bigcup\limits_{i=1, j\neq i}^{m} S_i(K) \cap S_j(K)$ is called a *critical set* of the system $\mathcal{S}$.

The violation of OSC is caused by overlaps of an attractor of system $\mathcal{S}$. If OSC does not hold, there is at least one point in C, but there is no guarantee that this point is unique - no such examples were constructed before.

Known methods like transversality method, which was used in [1], allow to construct the systems $\mathcal{S}_p$, depending of parameter p, such that $\mathcal{S}_p$ does not satisfy WSP (so as OSC) for Lebesgue-almost all p; but using this method we cannot control the type of overlaps.

Our method, based on General Position Theorem [2, Theorem 14], allows us to construct a families of self-similar sets with predictable behavior of critical set. For example, in [2] we get exact overlap for double fixed points. In the current work we prove the existence of system with unique one point intersection, not satisfying WSP.

We define a system $\mathcal{S}_{pq} = \{S_1, S_2, S_3, S_4\}$ of contraction similarities on $[0, 1]$ by the equations $S_1(x) = px$, $S_2(x) = h(q) - qx$, $S_3(x) = h(q) - \dfrac{1-x}{16}$, $S_4(x) = 1 - \dfrac{1-x}{16}$, where the contraction ratios $p, q \in (0, 1/16)$, and $h(q) = \dfrac{1+q}{1+16q}$. Let K_{pq} be the *attractor* of the system $\mathcal{S}_{pq}$.

From the construction it follows that $S_i(K_{pq}) \cap S_j(K_{pq}) \neq \varnothing$ with $i \neq j$ if and only if $\{i, j\} = \{3, 4\}$. We prove the following:

Theorem 1. *For Lebesgue-almost all* $(p, q) \in (0, 1/16)^2$*:*
(1) $C(S_{pq})$ *consists of one point* $h(q)$*;*
(2) $\mathcal{S}_{pq}$ *do not satisfy WSP.*

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References

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[2] K. Kamalutdinov, A. Tetenov, Twofold Cantor sets in $\mathbb{R}$. arXiv:1802.03872 1–13 (2018).