# On equitable 2-partitions of Hamming graphs $H(n, q)$ with eigenvalues $\lambda_2$

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A $r$-partition $C_1, \ldots, C_r$ of the vertex set of a graph is called *equitable* with quotient matrix $A = (A_{kl})_{k,l \in \{1,\ldots,r\}}$ if for any $k, l \in \{1, \ldots, r\}$ a vertex from $C_k$ has $A_{kl}$ neighbors in $C_l$. We treat equitable partitions as functions on the vertex sets, which take value $k$ on vertices from $C_k$ for $k \in \{1, \ldots, r\}$.

*An eigenvalue* of an equitable partition is an eigenvalue of its matrix of parameters. Any eigenvalue of an equitable partition of a graph is an eigenvalue of the graph, which is known as Lloyd theorem [2].

The Hamming graph $H(n, q)$ is the direct product of $n$ copies of the complete graph $K_q$. The eigenvalues of $H(n, q)$ are $\lambda_i(n) = (q - 1)n - qi$, $i \in \{0, \ldots, n\}$. Let the function $f' : V(H(n', q)) \to \{1, \ldots, r\}$ be obtained from an equitable $r$-partition $f$ of $H(n, q)$, $n < n'$ by adding nonessential coordinates: $f'(x_1, \ldots, x_{n'}) = f(x_1, \ldots, x_n)$. The function $f'$ is an equitable partition of $H(n', q)$ [3]. Moreover, $\lambda_i(n')$ is an eigenvalue of $f'$ iff $\lambda_i(n)$ is an eigenvalue of $f$. An equitable partition of $H(n, q)$ is called *reduced* if every its coordinate is essential.

The following characterization was obtained in [1]: the only reduced equitable 2-partitions of the Hamming graph $H(n, q)$ with eigenvalue $\lambda_1(n)$ are those with $n = 1$. Therefore the consideration of those with $\lambda_2(n)$ is naturally to be addressed.

**Construction A**. Let $q = 2t$ and consider a partition of $V(K_q)$ into complete graphs with vertex sets $V_1$ and $V_2$, $|V_1| = |V_2| = t$. Let $C_1$ be the following subset of vertices $H(4, q)$:

$$(V_1 \times V_1 \times V_1 \cup V_1 \times V_1 \times V_2 \cup V_2 \times V_1 \times V_2 \cup V_2 \times V_1 \times V_1) \times V_1 \cup$$

$$(V_2 \times V_2 \times V_1 \cup V_2 \times V_2 \times V_2 \cup V_2 \times V_1 \times V_2 \cup V_2 \times V_1 \times V_1) \times V_2.$$

**Proposition 1 [4].** *$C_1, \overline{C_1}$ is a reduced equitable 2-partition of $H(4, q)$ with eigenvalue $\lambda_2(n)$.*

**Construction B (Permutation switching construction)**. Consider a partition of $V(K_q)$ into $n - 1$ complete graphs with vertex sets $V_1, \ldots, V_{n-1}$. For any $i \in \{1, \ldots, n - 1\}$ let $f_i : V_i \times V(K_q) \to \{1, 2\}$ be an equitable 2-partition with eigenvalue $-2$ such that $\sum_{\alpha \in V(K_q)} f_i(x_1, \alpha) = c$, where $c$ is independent on the choices of $i$ in $\{1, \ldots, n\}$ and $x_1$ in $V_i$. Define the function $f : V(H(n, q)) \to \{1, 2\}$ as follows: for any $i \in \{1, \ldots, n - 1\}$ if $x_1$ in $V_i$ then for all $x_j \in V(K_q)$, $j \geq 3$ we have $f(x_1, x_2, \ldots, x_n) = f_i(x_1, x_2)$. The function $f$ is an equitable 2-partition of $H(n, q)$ with eigenvalue $\lambda_2(n)$. Further, the construction allows permutation switchings to be applied. Let $\pi_i$ be the transposition $(2, i+1)$. Define $\widehat{f} : V(H(n, q)) \to \{1, 2\}$ to be such that for any $i \in \{1, \ldots, n - 1\}$, $x_1$ in $V_i$ $\widehat{f}(x_1, \ldots, x_n) = f_i(\pi_i(x_1, \ldots, x_n))$.

**Proposition 2.** *The function $\widehat{f}$ is a reduced equitable 2-partition of $H(n, q)$ with the eigenvalue $\lambda_2(n)$.*

**Theorem.** *The only reduced equitable 2-partitions of $H(n, q)$ with eigenvalue $\lambda_2(n)$ are either reduced equitable partitions of $H(2, q)$ or $H(3, q)$ or the partitions obtained by constructions A or B.*

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## References

[1] A. D. Meyerowitz, Cycle-balance partitions for distance-regular graphs. *Discrete Math* **264:1-3** (2003) 149–165.

[2] D. M. Cvetkovic, M. Doob, H. Sachs, *Spectra of graphs*, Academic Press, New York, London, (1980).

[3] D. G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube.*Siberian Math. J* **48** (2007) 740–745.

[4] K. V. Vorob'ev, Alphabet lifting construction of equitable partitions of Hamming graphs. *Abstracts of G2R2* (2018) submitted.