Perfect 2-colorings of infinite circulant graphs with a continuous set of odd distances

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Consider an infinite graph $Ci_\infty(d_1, d_2, d_3, ..., d_n)$, whose set of vertices is the set of integers, and two vertices are adjacent if they are on the distance $d \in \{d_1, d_2, d_3, ..., d_n\}$. Let us call this construction an *infinite circulant graph.* A finite graph $Ci_t(d_1, d_2, d_3, ..., d_n)$ is a graph with the set of vertices coinciding with the set Z_t, for every vertex v having the multiset of incident edges $\{(v, v + d_i \bmod t)|i = 1, 2, ..., n\}$. There is a natural homomorphism from the set of vertices of the graph $Ci_\infty(d_1, d_2, d_3, ..., d_n)$ on the set of vertices of $Ci_t(d_1, d_2, d_3, ..., d_n)$ corresponding to the homomorphism from Z to Z_t.

Let k be a positive integer. A k*-coloring* of vertices of a graph $G = (V, E)$ is a map $\varphi : V \to \{1, 2, ..., k\}$. The value $\varphi(v)$ for a vertex $v \in V$ is called a *color* of v. A k-coloring of vertices is called *perfect*, if for every pair (i, j), where i, j are not necessarily distinct integers from the set $\{1, 2, \ldots, k\}$ there is an uniquely defined non-negative integer α_{ij} equals to the number of vertices of the color j in the neighborhood of each vertex of the color i. The *period* T of a coloring is a sequence $[\gamma_1\gamma_2...\gamma_t]$, where $\gamma_i = \varphi(v_{m+i})$ for an integer m, and $\varphi(v_l) = \varphi(v_{l+jt})$ for any choice of integers l and j. The coloring of a regular graph is uniquely defined by its period.

Perfect 2-colorings of circulant graphs are considered in [1, 2]. We are interested in so-called circulant graphs with a continuous set of odd distances, i.e. in ones with the property $d_i = 2i - 1, i = 1, 2, 3, ..., n$. Let us note, that for every positive integer n the graph $Ci_\infty(1, 3, 5, \ldots, 2n - 1)$ is regular of degree $2n$ and is bipartite.

We state the following conjecture.

Conjecture. *Let k and n be positive integers. The set of perfect k-colorings of a graph $Ci_\infty(1, 3, 5, \ldots, 2n - 1)$ consists of all perfect k-colorings of graphs $Ci_t(1, 3, 5, \ldots, 2n - 1)$ for $t = 4n-2, 4n, 4n+2$ and the following four:* $[123...k]$, $[123...(k-1)k(k-1)...32]$, $[123...(k-1)kk(k-1)...32]$, $[123...(k-1)kk(k-1)...321]$.

We are going to present the proof of this conjecture in case of $k = 2$ and for arbitrary n. A similar conjecture for infinite circulants $Ci_\infty(1, 2, \ldots, n), n \in \mathbb{N}$ was posed in [3]. It was proved in case $k = 2$ and $n \in \mathbb{N}$ in [1], and in case $n = 2$ for arbitrary k in [4]. The natural homomorphism from n-dimensional grid Z^n on $Ci_\infty(1, 2, \ldots, n)$ shows that the problem of complete description of perfect colorings in arbitrary number of colors of such graphs is rather complicated.

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References

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