## A simple multibody system on a discrete circle

Chengyang Qian
*Shanghai Jiao Tong University, Shanghai, China*
mike98(at)sjtu.edu.cn

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Let $n$ be a positive integer and let $\mathbb{Z}_n$ denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of residue classes of integers modulo $n$, and let the integers $i \in \mathbb{Z}$ also denote their residue classes $i + n\mathbb{Z} \in \mathbb{Z}_n$ as long as no confusion can result. For each subset $A$ of $\mathbb{Z}_n$, we call $i \in \mathbb{Z}_n$ a head of $A$ provided $i \in A$ and $i-1 \notin A$, and we call $i \in \mathbb{Z}_n$ a tail of $A$ provided $i \in A$ and $i+1 \notin A$. We call a subset of $\mathbb{Z}_n$ a proper interval if it has a unique head $i$ and a unique tail $j$ and we designate this proper interval by $[i,j]_n$. We often use $[j]_n$ for $[1,j]_n$.

We play a roulette game on a discrete circle with $n$ slots. The $n$ slots placed circularly can be naturally identified with $\mathbb{Z}_n$. We choose a positive integer $k$ satisfying $2 \leq k \leq n$ and put $k$ undistinguished balls on $k$ slots of the circle. Suppose that the slots of the $k$ balls are read as $a_1, \ldots, a_k \in \mathbb{Z}_n$ such that the interval with head $a_i$ and tail $a_{i+1}$ contains exactly two balls, one at slot $a_i$ and one at slot $a_{i+1}$, for $i = 1, \ldots, k-1$. In one step movement, the ball at slot $a_i$ can move to any slot from the interval $[a_i, a_{i+1}-1]_n$ with equal probability. This gives an ergodic Markov chain with $\binom{n}{k}$ states. The support of this Markov chain is an Eulerian digraph $\mathcal{R}_{n,k}$ of diameter $k$ and with $\mathbb{Z}_n$ as its automorphism group. The probability of visiting any state in the stationary distribution of the Markov chain is proportional to the degree of the state in this Eulerian digraph.

We study the linear algebra around the above simple $k$-body problem on a discrete circle. We use $R_{n,k}$ for the linear map from $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ to itself that sends $A \in \binom{\mathbb{Z}_n}{k}$ to $\sum B$ where $B$ runs through all out-neighbors of $A$ in the digraph $\mathcal{R}_{n,k}$; we also use $R_{n,k}$ for the matrix of the corresponding linear map with respect to the basis $\binom{\mathbb{Z}_n}{k}$. We show that $R_{n,k}$ has rank $\binom{n}{k}$ when $k$ is even and $\binom{n-1}{k-1}$ when $k$ is odd. We prove that the extremal rays of the cone generated by the rows of $R_{n,k}$ are just all its rows. When $k$ is even, we can also determine all facets of this cone. We list part of these results in a more precise way below.

Assume that $1 < k < n$. For $A \in \binom{\mathbb{Z}_n}{k}$ and $B \in \binom{[n-1]_n}{k}$, put

$$f_n(A) := \sum_{j \in \mathbb{Z}_n \setminus A} (-1)^{|[j]_n \cap A|}(\{j\} \cup A) \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$$

and

$$g_n(B) := B + \sum_{j \in B} (-1)^{|[j]_n \cap B|}((B \setminus \{j\}) \cup \{n\}) \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}.$$

For $C = \{c_1, \ldots, c_k\} \in \binom{\mathbb{Z}_n}{k}$ and $\epsilon \in \{0,1\}^C$, let $C_\epsilon := \{c_1 - \epsilon_{c_1}, \ldots, c_k - \epsilon_{c_k}\} \in \left(\!\binom{\mathbb{Z}_n}{k}\!\right)$, and

$$h_n(C) := \sum_{\epsilon \in \{0,1\}^C, C_\epsilon \in \binom{\mathbb{Z}_n}{k}} (-1)^{\sum_{i \in C} \epsilon_i} C_\epsilon \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}.$$

We view $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ as an Euclidean space with $\binom{\mathbb{Z}_n}{k}$ as a standard normal basis and we write $\langle \cdot, \cdot \rangle$ for the corresponding inner product. The next result gives an expression of the inverse matrix of $R_{n,k}$ when $k$ is even.

**Theorem 1.** *When $k$ is even and $1 < k < n$, $\langle R_{n,k}(A), h_n(C) \rangle = 2\delta_{A,C}$ for $A, C \in \binom{\mathbb{Z}_n}{k}$.*

The gap of a state $C \in \binom{\mathbb{Z}_n}{k}$, denoted by $\|C\|$, is given by

$$\|C\| := \min_{(a,b) \in C \times C, a \neq b} |[a,b]_n| - 1.$$

Let $R_{n,k,t}$ denote the submatrix of $R_{n,k}$ obtained by removing all rows indexed by $\{C : \|C\| < t\}$. Note that $R_{n,k,1} = R_{n,k}$. Here is an easy corollary of Theorem 1.

**Corollary 1.** *When $k$ is even and $1 < k < n$, $\varphi \in \mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ is a nonnegative linear combination of the rows of $R_{n,k,t}$ if and only if its inner product with $h_n(C)$ is nonnegative when $\|C\| \geq t$ and is 0 when $\|C\| < t$ for all $C \in \binom{\mathbb{Z}_n}{k}$.*

When $(k,t) = (2,1)$, Corollary 1 reduces to the characterization of Kalmanson matrix [2, Theorem 33]; when $(k,t) = (2,2)$, Corollary 1 reduces to the characterization of Kalmanson metrics [2, Theorem 31] [1, Theorem] [3, Theorem 5.2].

**Theorem 2.** *Assume that $k$ is odd and $1 < k < n$. Then, $\left\{f_n(A) :\ A \in \binom{[n-1]_n}{k-1}\right\}$ forms a basis of $\operatorname{Im} R_{n,k}$, while both $\left\{g_n(B) :\ B \in \binom{[n-1]_n}{k}\right\}$ and $\left\{h_n(C) :\ C \in \binom{[n-1]_n}{k}\right\}$ are bases of $\operatorname{Ker} R_{n,k}$. Moreover, $\operatorname{Im} R_{n,k}$ and $\operatorname{Ker} R_{n,k}$ are orthogonal complements of each other in $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$.*

The study of $R_{n,2}$ is related to trees, consecutive-ones property, circular split systems, Kalmanson matrics and Robinsonian matrices. In this ongoing research, we intend to see what happens for general $k$.

Suppose that $a_1, \ldots, a_k \in \mathbb{Z}_n$ are $k$ different elements such that the interval with head $a_i$ and tail $a_{i+1}$ contains exactly two of these $k$ elements, namely $a_i$ and $a_{i+1}$ for $i = 1, \ldots, k-1$. Define $\mathrm{E}_{n,k}(a_1 \wedge \cdots \wedge a_k) = (\sum_{a \in [a_1, a_2-1]} a) \wedge \cdots \wedge (\sum_{a \in [a_{k-1}, a_k-1]} a) \wedge (\sum_{a \in [a_k, a_1-1]} a)$. This induces a well-defined linear map $\mathrm{E}_{n,k}$ from $\wedge^k(\mathbb{R}^{\mathbb{Z}_n})$ to itself. Let $\Theta_n^k$ be the linear map from $\mathbb{R}^{\binom{\mathbb{Z}_n}{k}}$ to $\bigwedge^k \mathbb{R}^{\mathbb{Z}_n}$ such that for all $C \in \binom{\mathbb{Z}_n}{k}$, $\Theta_n^k(C) := c_1 \wedge \cdots \wedge c_k$, where $C = \{c_1, \ldots, c_k\}$ and $c_i \in [c_{i+1} - 1]_n$ for all $i = 1, \ldots, k-1$. Given $C \in \binom{\mathbb{Z}_n}{k}$, let $h''_n(C)$ be

$$\sum_{\epsilon \in \{0,1\}^C, C_\epsilon \in \binom{\mathbb{Z}_n}{k}} (-1)^{(k-1)\epsilon_n + \sum_{i \in C} \epsilon_i} \Theta_n^k(C_\epsilon) \in \wedge^k \mathbb{R}^{\mathbb{Z}_n}.$$

provided $n \in C$, and let $h''_n(C)$ be

$$\sum_{\epsilon \in \{0,1\}^C, C_\epsilon \in \binom{\mathbb{Z}_n}{k}} (-1)^{\sum_{i \in C} \epsilon_i} \Theta_n^k(C_\epsilon) \in \wedge^k \mathbb{R}^{\mathbb{Z}_n}$$

otherwise.

**Theorem 3.** *Both $\{\Theta_n^k \circ f_n(A) :\ A \in \binom{[n-1]_n}{k-1}\}$ and $\{\mathrm{E}_{n,k} \circ \Theta_n^k(A \cup \{n\}) :\ A \in \binom{[n-1]_n}{k-1}\}$ are bases of $\operatorname{Im} \mathrm{E}_{n,k}$; while both $\{\Theta_n^k \circ g_n(B) :\ B \in \binom{[n-1]_n}{k}\}$ and $\{h''_n(C) :\ C \in \binom{[n-1]_n}{k}\}$ are bases of $\operatorname{Ker} \mathrm{E}_{n,k}$.*

## References


[1] V. Chepoi, B. Fichet, A note on circular decomposable metrics. *Geom. Dedicata* **69**(3) (1998) 237–240.

[2] G. Christopher, M. Farach, M. Trick, The structure of circular decomposable metrics. In *Algorithms—ESA '96 (Barcelona)*, volume 1136 of *Lecture Notes in Comput. Sci.* 486–500. Springer, Berlin, 1996.

[3] A. Kleinman, M. Harel, L. Pachter. Affine and projective tree metric theorems. *Annals of Combinatorics* **17**(1) (2013) 205–228.