Competition Numbers and Phylogeny Numbers

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A *graph* G is a pair consisting of its vertex set $\mathrm{V}(G) \neq \emptyset$ and its edge set $\mathrm{E}(G) \subseteq \binom{\mathrm{V}(G)}{2}$. For each graph G and nonnegative integer k, let $\mathrm{I}_k(G)$ stand for the graph obtained from G by adding k isolated vertices. A *vertex-induced subgraph* of a graph G, or simply known as a subgraph of G, is a graph G' such that $\mathrm{V}(G') \subseteq \mathrm{V}(G)$ and $\mathrm{E}(G') = \mathrm{E}(G) \cap \binom{\mathrm{V}(G')}{2}$. Let us write $G' \lhd G$ to mean that G' is a subgraph of G. A digraph D is a pair consisting of its vertex set $\mathrm{V}(D) \neq \emptyset$ and its arc set $\mathrm{A}(D) \subseteq \mathrm{V}(D) \times \mathrm{V}(D)$. For each digraph D, let D° stand for the digraph with $\mathrm{V}(D^\circ) = \mathrm{V}(D)$ and $\mathrm{A}(D^\circ) = \mathrm{A}(D) \cup \{(v, v) : \ v \in \mathrm{V}(D)\}$. For any $(u, v) \in \mathrm{A}(D)$, we call u an *in-neighbor* of v in D, and call v an *out-neighbor* of u in D. A digraph D is *acyclic* if it contains no cycle.

For every digraph D, the *competition graph* of D [1], denoted by $\mathrm{C}(D)$, is the graph with $\mathrm{V}(\mathrm{C}(D)) = \mathrm{V}(D)$ and with two vertices being adjacent if and only if they have at least one common out-neighbor in D. The *competition number* of a graph G, denoted by $\kappa(G)$, is the least nonnegative integer k such that $\mathrm{I}_k(G)$ becomes the competition graph of an acyclic digraph. Equivalently, $\kappa(G) = \min(|\mathrm{V}(D)| - |\mathrm{V}(G)|)$ where D runs through all acyclic digraphs such that $G \lhd \mathrm{C}(D)$.

For every digraph D, the *phylogeny graph* of D [4], denoted by $\mathcal{P}(D)$, is the competition graph of D°, that is, $\mathcal{P}(D) = \mathrm{C}(D^\circ)$. Note that phylogeny graphs are known as moral graphs in Bayesian network theory [3]. The *phylogeny number* of a graph G, denoted by $\phi(G)$, is the least number p such that we can find a phylogeny graph of an acyclic digraph that has $p + |\mathrm{V}(G)|$ vertices and has G as an induced subgraph.

A *hypergraph* H comprises its vertex set $\mathrm{V}(H) \neq \emptyset$ and its hyperedge set $\mathcal{E}(H) \subseteq \binom{\mathrm{V}(H)}{\geq 2}$. For each hypergraph H and nonnegative integer k, let $\mathrm{I}_k(H)$ stand for the hypergraph with $\mathcal{E}(\mathrm{I}_k(H))$ equals $\mathcal{E}(H)$ and $\mathrm{V}(\mathrm{I}_k(H)) \setminus \mathrm{V}(H)$ is a set of size k. The *subhypergraph* induced by a nonempty subset $A \subseteq \mathrm{V}(H)$ is the hypergraph H' with vertex set A and hyperedge set $\mathcal{E}(H') = \{e \cap A : \ e \in \mathcal{E}(H), e \cap A \neq \emptyset\}$. For two hypergraphs H and H', we write $H' \lhd H$ to mean that H' is a subhypergraph of H. For every digraph D, the *competition hypergraph* of D [5], denoted by $\mathcal{CH}(D)$, is the hypergraph with vertex set $\mathrm{V}(\mathcal{CH}(D)) = \mathrm{V}(D)$ and hyperedge set

$$\mathcal{E}(\mathcal{CH}(D)) = \{e \in \binom{\mathrm{V}(H)}{\geq 2} : \ \exists v \in \mathrm{V}(D) \text{ s.t. } e = \{w : \ (w, v) \in \mathrm{A}(D)\}\}.$$

The ST-*competition number of a hypergraph* H, denoted by $\kappa_{\text{ST}}(H)$, is the least nonnegative integer k such that $\mathrm{I}_k(H)$ becomes the competition hypergraph of an acyclic digraph. Equivalently, $\kappa_{\text{ST}}(H)$ is the least value of $|\mathrm{V}(D) \setminus \mathrm{V}(H)|$ where D runs through all acyclic digraphs satisfying $H \lhd \mathcal{CH}(D)$.

For every digraph D, the ST-*phylogeny hypergraph* of D, denoted by $\mathcal{PH}(D)$, is the competition hypergraph of D°, that is, $\mathcal{PH}(D) = \mathcal{CH}(D^\circ)$. The ST-*phylogeny number of a hypergraph* H, which we write as $\phi_{\text{ST}}(H)$, is the least value of $|\mathrm{V}(D) \setminus \mathrm{V}(H)|$ where D runs through all acyclic digraphs satisfying $H \lhd \mathcal{PH}(D)$.

Theorem 1. *The ranges of the functions $\phi - \kappa + 1$ and $\phi_{ST} - \kappa_{ST} + 1$ are both the set of nonnegative integers.*

Theorem 2. *For any two hypergraphs H_1 and H_2, it holds $\phi_{ST}(H_1 \sqcup H_2) = \phi_{ST}(H_1) + \phi_{ST}(H_2)$, where $H_1 \sqcup H_2$ stands for the disjoint union of H_1 and H_2.*

For any positive integers $m, n_1, \dots, n_m$, let $[m] = \{1, \dots, m\}$ and let $K^{n_1,\dots,n_m}$ denote the graph with

$$\mathrm{V}(K^{n_1,\dots,n_m}) = \bigcup_{i=1}^{m} \mathrm{V}_i$$

where $\mathrm{V}_i = \{v_i^j : j \in [n_i]\}$ for $i \in [m]$, and with

$$\mathrm{E}(K^{n_1,\dots,n_m}) = \{v_i^j v_{i'}^{j'} : i \neq i', j \in [n_i], j' \in [n_{i'}]\}.$$

We call $K^{n_1,\dots,n_m}$ a *complete multipartite graph* with m parts and part size $n_1, \dots, n_m$. The *uniform complete multipartite graph*, denoted by K_m^n, is the complete multipartite graph $K^{n,\dots,n}$ with m parts and uniform part size n.

Theorem 3.

(1) $\phi(K_m^2) - \kappa(K_m^2) + 1 = 0$ *for* $m \geq 2$*;*

(2) $\phi(K_m^3) - \kappa(K_m^3) + 1 = 0$ *for* $m \geq 3$*;*

(3) $\phi(K_3^n) - \kappa(K_3^n) + 1 = 0$ *for* $n \geq 2$*.*

For a graph G, a *clique* of G is a subset of $\mathrm{V}(G)$ such that every two vertices in this subset are adjacent. A clique of G is called *maximal* if it is not properly contained in every clique of G. The *clique hypergraph* of G, denoted by $\mathcal{K}(G)$, is the hypergraph with vertex set $\mathrm{V}(G)$ and with the set of all maximal cliques of G as its hyperedge set. For $G = K^n$, it is easy to see that $\phi_{\mathrm{ST}}(\mathcal{K}(G)) = \kappa_{\mathrm{ST}}(\mathcal{K}(G)) = 0$.

Theorem 4. *Let* $m, n_1, \dots, n_m$ *be positive integers and let* $H = \mathcal{K}(K^{n_1,\dots,n_m})$*. If* $m \geq 2$*, then* $\phi_{ST}(H) + 1 = \kappa_{ST}(H) = \prod_{\ell=1}^{m} n_\ell - \sum_{\ell=1}^{m} n_\ell + m$*.*

A number of Latin squares of the same order form a set of *mutually orthogonal Latin squares*, often abbreviated in the literature to MOLS, if any two of them are orthogonal. The largest size of a set of MOLS of order n is denoted by $\mathcal{L}(n)$.

Theorem 5. *(Kim-Park-Sano [2, Theorem 3]) Let* m *and* n *be integers such that* $3 \leq n = \mathcal{L}(n) + 1 \leq m$*. Then* $\kappa(K_m^n) \leq n^2 - n + 1$*.*

Theorem 6. *Let* n *be a positive integer such that* $\mathcal{L}(n) = n - 1$*. Then for every integer* m *bigger than 1, it holds* $\kappa(K_m^n) \leq n^2 - 2n + 2$*.*

References

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