Tropical hyperplane arrangements and zonotopal tilings

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This is joint work with Yaokun Wu

In tropical geometry, people study the tropical semiring consisting of the real numbers with operations of min and +. It has received considerable attention recently due to the strong relationship between several classical questions and their tropical counterparts. Tropical polytopes and tropical hyperplane arrangements have been widely studied [1–3, 6].

Zonotopes are fundamental objects in combinatorial geometry that are closely related to hyperplane arrangements, oriented matroids, and tilings [8,9]. In this paper, we want to study the zonotopal structure arising from tropical hyperplane arrangements. It turns out that zonotopal tropical complexes have nice connections with tree metrics and tropical lines.

For each element $v \in \mathbb{R}^n$, we let $[v]$ denote the equivalence class of v in the tropical projective torus $TT^{n-1} = \mathbb{R}^n/\mathbb{R}\mathbf{1}$ where $\mathbf{1}$ denotes the "all-ones" vector in $\mathbb{R}^n$. These equivalence classes will also be called the "points"of TT^{n-1}. The tropical hyperplane in TT^{n-1} with apex $[v]$ is the set of elements $[x] \in TT^{n-1}$ for which the maximum in $\{x_1 - v_1, \ldots, x_n - v_n\}$ is achieved at least twice. Each tropical hyperplane induces a subdivision of TT^{n-1}. Given a discrete set V in TT^{n-1}, we have a tropical hyperplane arrangement whose apices are exactly the points in V. It induces a subdivision of TT^{n-1} that is defined as the common refinement of the subdivisions of TT^{n-1} induced by the tropical hyperplanes in that arrangement, and we denote the subcomplex of bounded faces of this subdivision by $\mathcal{C}_V$. Develin and Sturmfels [3] call $\mathcal{C}_V$ the tropical complex generated by V and observe that the union of faces in $\mathcal{C}_V$ coincides with the tropical convex set generated by V.

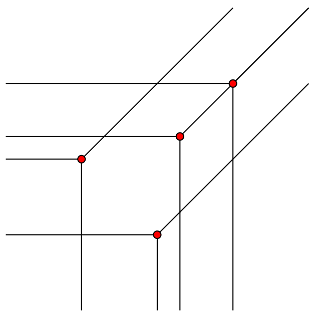

Рис. 1: A tropical hyperplane arrangement in TT^2.

A zonotope is an affine projection of a hypercube. A zonotopal tiling is a subdivision of a polytope by zonotopes and a zonotopal filling is a subdivision of the whole Euclidean space by zonotopes. In Fig. 1, we depict a tropical hyperplane arrangement whose tropical complex is a zonotopal tiling.

Tropical linear spaces are those polyhedral complexes that satisfy the balance condition (also called the zero-tension condition) [5, 7]. We call one-dimensional tropical linear spaces tropical lines. Each tropical line has the shape of an infinite tree.

Theorem 1. *Let V be a finite subset of TT^{n-1}. Then $\mathcal{C}_{-V}$ is a zonotopal tiling if and only if V is on a tropical line T and each ray of T contains at least one point from V.*

Theorem 2. *Let V be an infinite discrete subset of TT^{n-1}. Then $\mathcal{C}_{-V}$ is a zonotopal filling of TT^{n-1} if and only if V is on a tropical line T and each ray of T contains infinitely many points from V.*

Suppose V is a finite subset of a tropical line T such that each ray of T contains at least one point from V. For each ray R of T, we let v_R be the outmost point of R that is in V. We call $Ext(V) := \{v_R \in V : R \text{ is a ray of } T\}$ the extremal point set of V. Let $conv_T(V)$ be the minimum subtree of T containing V. Define T_V to be the subdivision of $conv_T(V)$ by regarding points in V as vertices (dimension zero faces). A partial edge orientation of T_V is called a *cyclic orientation* if all sink and source vertices are in V and every vertex in $V \setminus Ext(V)$ is on an outgoing arc. Define $CO(T_V)$ to be the poset on all cyclic orientation of T_V ordered under inclusion. See Fig. 2 for an example.

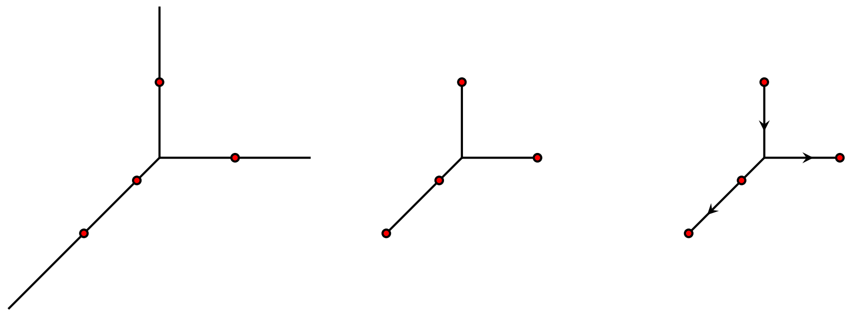

Рис. 2: A cyclic orientation of T_V.

Theorem 3. *Let V be a finite set in TT^{n-1}. If $\mathcal{C}_{-V}$ is a zonotopal tiling, then the face poset of $\mathcal{C}_{-V}$ is anti-isomorphic with $CO(T_V)$, namely there is a bijection α from the face poset of $\mathcal{C}_{-V}$ to $CO(T_V)$ such that A is contained in B if and only if $\alpha(B)$ is contained in $\alpha(A)$ for all faces A and B of $\mathcal{C}_{-V}$.*

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