Relative t-designs on one shell of Johnson association schemes

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The concept of relative t-designs in Q-polynomial association schemes was first introduced by Delsarte [1] in 1977. For any fixed point $u_0 \in \binom{V}{k}$, each nontrivial shell of Johnson association scheme $J(v,k) = (\binom{V}{k}, \{R_r\}_{0\le r\le k})$ w.r.t. u_0 is known to be a commutative association scheme which is the product of two smaller Johnson association schemes, but it is not a Q-polynomial association scheme anymore. (The r-th shell of $J(v,k)$ is defined by $X_r = \{x \in \binom{V}{k} : |x \cap u_0| = k - r\}$)

In 1999, Martin [3] defined t-designs in the product of Q-polynomial association schemes. In particular, they are called mixed t-designs in [2] if they are defined on the product of two Johnson association schemes. Moreover, he gave a combinatorial interpretation of Delsarte's generalized Assmus-Mattson Theorem for the Johnson association schemes. We generalize the result of Martin [2] by weakening the assumption of t-designs to relative t-designs and obtain the following result.

Theorem 1. *Fix a point $u_0 \in \binom{V}{k}$. Let Y be a subset of $S := X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$ and $w : Y \to \mathbb{R}_{>0}$ be a weight function on Y. If (Y,w) is a relative t-design in $J(v,k)$ with respect to u_0 on p shells S, then $(Y \cap X_{r_\nu}, w)$ is a weighted $(t+1-p)$-design in X_{r_ν} for $1 \le \nu \le p$.*

In the theorem above, weighted t-designs in X_r are the weighted generalization of mixed t-designs.

We also study the classification problems of tight 2-, 3- and 4-designs in one shell X_r of Johnson association schemes $J(v,k)$ with small parameters, say $v \le 1,000$. (Tight t-designs are those whose size attains the lower bound)

(1) If $t = 2$, one interesting observation is that, for $\frac{k}{2} \le r \le \frac{v-k}{2}$, we can expect that all such tight objects come from symmetric 2-designs. However, there is one exception with $v = 528$ which we cannot eliminate now.

(2) If $t = 3$, all the known examples with $v \le 1,000$ arise from Hadamard 2-$(4u-1, 2u-1, u-1)$ designs, but it is still open to prove this theoretically.

(3) There exists no tight relative 4-design on one shell for $v \le 1,000$ according to the result of computer search.

References

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[3] W. J. Martin, Designs in product association schemes. *Designs, Codes and Cryptography* **16** (1999) 271–289.