On the splitness of the prime and solvable graphs for finite simple groups

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A graph is **split** if its vertex set can be partitioned into a clique and an independent set (either set can be empty). Split graphs were introduced independently by Földes and Hammer [1], and by Tyshkevich and Chernyak [2]. Split graphs are a popular subclass of chordal graphs having many interesting properties. In particular, as proved in [3], split graphs can be characterized in terms of their degree sequences.

We consider two similar types of graphs associated with finite groups. Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of $|G|$. The **prime graph** $\mathrm{GK}(G)$ (the **solvable graph** $\mathcal{S}(G)$) of G is a graph with the vertex set $\pi(G)$, in which two different vertices r and s are adjacent if and only if G has a *cyclic* (*solvable*) subgroup of order divisible by rs.

The prime and solvable graphs of finite simple groups are well studied, see, e.g., [4] and [5]. Relying on these results, we investigate splitness of these graphs. If G is an abelian simple group, then $\mathrm{GK}(G)=\mathcal{S}(G)$ is a singleton, and so is split. Thus, we may assume that G is a nonabelian simple group. Easy calculations show that if G is an alternating group, then the graphs $\mathrm{GK}(G)$ and $\mathcal{S}(G)$ are split. Using [6], we check that the prime graph $\mathrm{GK}(G)$ of any sporadic group G is split, and that there are exactly 10 sporadic groups whose solvable graph $\mathcal{S}(G)$ is nonsplit.

Let G be a finite simple group of Lie type. It is easy to see that in most cases the prime graph $\mathrm{GK}(G)$ is not split. However, it turns out that the **compact form** $\mathrm{GK}_{\mathrm{c}}(G)$ of this graph is split. Here by the compact form Γ_{c} of a graph Γ we mean the quotient graph $\Gamma/_{\equiv}$ with respect to the following equivalence relation on the vertex set V_Γ of Γ: we put $u \equiv v$ if and only if u and v are equal or adjacent and have the same neighbourhood for every $u, v \in V_\Gamma$. Note that the compact form of a split graph is split. Our main result is as follows.

Theorem. *The graph* $\mathrm{GK}_{\mathrm{c}}(G)$ *of any finite simple group* G *is split.*

In general, the compact form $\mathcal{S}_{\mathrm{c}}(G)$ of the solvable graph for a simple group of Lie type also splits. Nevertheless, there are examples of nonsplitness of such graphs, and due to the positive solution of the Artin conjecture on primitive roots for almost all primes [7], one can construct infinitely many of them.

References

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