Problem Solving Session 1

Problem A

Let n be an integer greater than 4, and consider the action of the symmetric group $S_n = \mathrm{Sym}(\{1, \ldots, n\})$ on the set

$$\Omega = \{\{i, j\} \mid i, j \in \{1, \ldots, n\},\ i \neq j\}.$$

(i) Find the set $\mathcal{R} = \{R_0, R_1, R_2\}$ of all 2-orbits of this action.

(ii) Find all the intersection numbers p_{ij}^k defined by

$$A_i A_j = \sum_{h=0}^{2} p_{ij}^k A_k,$$

where $A_k = A(R_k)$ for $k = 0, 1, 2$.

(iii) Find the minimal polynomial of A_k for $k = 0, 1, 2$.

(iv) Find the spectrum of A_k for $k = 0, 1, 2$.

Problem B

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v elements, $\mathcal{B}$ is a family of k-element subsets of $\mathcal{P}$ such that

$$\forall T \subseteq \mathcal{P} \text{ with } |T| = t,\ |\{B \in \mathcal{B} \mid T \subseteq B\}| = \lambda.$$

It is known that there exists a 4-$(23, 7, 1)$ design called the Witt system $W_{23} = (\mathcal{P}, \mathcal{B})$, whose uniqueness was proved by E. Witt in 1938. In the following, I and J denote disjoint subsets of $\mathcal{P}$ with $|I| = i$ and $|J| = j$. Define

$$\lambda_I^J = |\{B \in \mathcal{B} \mid I \subseteq B,\ J \cap B = \emptyset\}|.$$

(i) Show that, for all $i \leq 4$, $\lambda_I^\emptyset$ depends only on i, and is independent of the choice of I. Denote these numbers by λ_i instead, and compute them for $i \leq 4$.

(ii) Show that, for all i, j with $i + j \leq 4$, λ_I^J depends only on i and j, and is independent of the choice of I and J. Denote these numbers by λ_i^j instead, and compute them for $i + j \leq 4$.

(iii) Assume $I \cup J \subseteq B$ for some $B \in \mathcal{B}$. Show that, for all i, j with $5 \leq i + j \leq 7$, λ_I^J depends only on i and j, and is independent of the choice of I and J. Denote these numbers by λ_i^j instead, and compute them for $5 \leq i + j \leq 7$.

(iv) Deduce that for any distinct $B_1, B_2 \in \mathcal{B}$,

$$|B_1 \cap B_2| = 1 \text{ or } 3.$$

Problem Solving Session 2

Problem C

The Mathieu group M_{23}, discovered by É. Mathieu in 1873 is a subgroup of the symmetric group S_{23} satisfying the following property: for any two ordered quadruples $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)$, each consisting of four distinct elements of $\mathcal{P} = \{1, 2, \ldots, 23\}$, there exists a permutation $\sigma \in M_{23}$ such that

$$(x_1^\sigma, x_2^\sigma, x_3^\sigma, x_4^\sigma) = (y_1, y_2, y_3, y_4).$$

It is also known that M_{23} is the automorphism group of the Witt system $W_{23} = (\mathcal{P}, \mathcal{B})$. The 2-orbits of M_{23} on $\mathcal{B}$ are

$$\begin{aligned}
R'_0 &= \{(B_1, B_1) \mid B_1 \in \mathcal{B}\}, \\
R'_1 &= \{(B_1, B_2) \mid B_1, B_2 \in \mathcal{B},\ |B_1 \cap B_2| = 3\}, \\
R'_2 &= \{(B_1, B_2) \mid B_1, B_2 \in \mathcal{B},\ |B_1 \cap B_2| = 1\}.
\end{aligned}$$

Let

$$\Omega = \{\{i, j\} \mid i, j \in \{1, \ldots, 23\},\ i \neq j\},$$

and consider the action $(M_{23}, \Omega \cup \mathcal{B})$.

(i) Use Problem B(iii) to compute $|R'_1|$ and $|R'_2|$.

(ii) Show that there are at least three 2-orbits contained in $\Omega \times \mathcal{B}$. Deduce that there are at least twelve 2-orbits in this action.

(iii) Show that the dimension of the centralizer algebra $V_{\mathbb{C}}(M_{23}, \Omega \cup \mathcal{B})$ is at most 12.

(iv) Deduce that the two permutation representations of M_{23} on Ω and $\mathcal{B}$ are equivalent as representations.

(v) Compare (i) and Problem A(ii) to deduce that the two permutation representations of M_{23} on Ω and $\mathcal{B}$ are not equivalent as permutation representations.