Representations of the virtual braid group

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Braid group B_n on $n \geq 2$ strands is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and is defined by relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2, \tag{1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2. \tag{2}$$

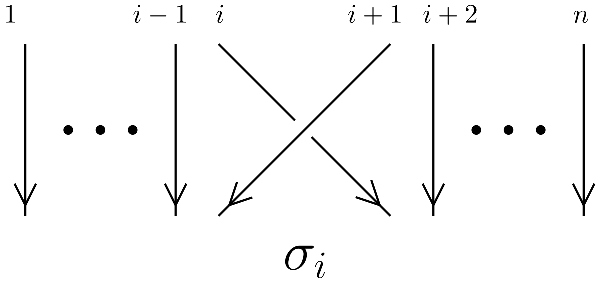

Figure: Geometric interpretation of σ_i

The Artin representation

$$\varphi_A : B_n \longrightarrow \mathrm{Aut}(F_n),$$

where $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is a free group, is defined by the rule

$$\varphi_A(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \end{cases}$$

Here and onward we point out only nontrivial actions on generators assuming that other generators are fixed.

Theorem [Artin]: $Ker(\varphi_A) = 1$.

Let $\mathcal{L}$ be the set of all links in $\mathbb{R}^3$.

A group $G(L)$ of a link $L \in \mathcal{L}$ is a group $\pi_1(\mathbb{R}^3 \setminus L)$.

Theorem [Artin]: If L is isotopic to $\hat{\beta}$, where $\beta \in B_n$, then

$$G(L) = \langle x_1, x_2, \ldots, x_n \parallel x_i = \varphi_A(\beta)(x_i), \quad i = 1, 2, \ldots, n \rangle.$$

The virtual braid group VB_n is presented by L. Kauffman (1996).
V. Vershinin constructed the more compact system of defining relations for VB_n.

VB_n is generated by the classical braid group $B_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ and the permutation group $S_n = \langle \rho_1, \dots, \rho_{n-1} \rangle$. Generators $\rho_i, i = 1, \dots, n-1$, satisfy the following relations:

$$\rho_i^2 = 1 \qquad \text{for} \quad i = 1, 2, \dots, n-1, \tag{3}$$

$$\rho_i \rho_j = \rho_j \rho_i \qquad \text{for} \quad |i-j| \geq 2, \tag{4}$$

$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \qquad \text{for} \quad i = 1, 2 \dots, n-2. \tag{5}$$

Other defining relations of the group VB_n are mixed and they are as follows

$$\sigma_i \rho_j = \rho_j \sigma_i \qquad \text{for} \quad |i-j| \geq 2, \tag{6}$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} \qquad \text{for} \quad i = 1, 2, \dots, n-2. \tag{7}$$

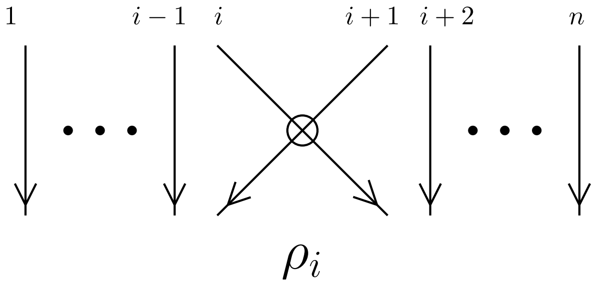

Figure: Geometric interpretation of ρ_i

- Construct a faithfull representation

$$\psi : VB_n \longrightarrow \mathrm{Aut}(H),$$

where H is a “good” group.

- Define a group of virtual link.

We consider the free product $F_{n,2n+1} = F_n * \mathbb{Z}^{2n+1}$, where F_n is a free group of the rank n generated by elements $x_1, x_2, \ldots, x_n$ and $\mathbb{Z}^{2n+1}$ is a free abelian group of the rank $2n+1$ freely generated by elements $u_1, u_2, \ldots, u_n$, $v_0, v_1, v_2, \ldots, v_n$.

Theorem 1 [V. B. – Yu. Mikhalchishina – M. Neshchadim, 2017].

The following mapping $\varphi_M : VB_n \longrightarrow \mathrm{Aut}(F_{n,2n+1})$ is defined by the action on the generators:

$$\varphi_M(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1}^{u_i} x_i^{-v_0 u_{i+1}}, \\ x_{i+1} \longmapsto x_i^{v_0}, \end{cases} \qquad \varphi_M(\sigma_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i, \end{cases}$$

$$\varphi_M(\sigma_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\varphi_M(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \longmapsto x_i^{v_{i+1}}, \end{cases} \qquad \varphi_M(\rho_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i, \end{cases}$$

$$\varphi_M(\rho_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

is provided a representation of VB_n into $\mathrm{Aut}(F_{n,2n+1})$, which generalizes all known representations.

The constructed representation φ_M is not an extension of the Artin representation.

It is turned out that the representation φ_M is equivalent to the simpler one which is an extension of the Artin representation.

Let $F_{n,n} = F_n * \mathbb{Z}^n$, where $F_n = \langle y_1, y_2, \ldots, y_n \rangle$ is the free group and $\mathbb{Z}^n = \langle v_1, v_2, \ldots, v_n \rangle$ is the free abelian group of the rank n.

Theorem 2 [V. B. – Yu. Mikhalchishina – M. Neshchadim, 2017].

The representation $\widetilde{\varphi}_M : VB_n \longrightarrow \mathrm{Aut}(F_{n,n})$ defined by the action on the generators

$$\widetilde{\varphi}_M(\sigma_i) : \begin{cases} y_i \longmapsto y_i y_{i+1} y_i^{-1}, \\ y_{i+1} \longmapsto y_i, \end{cases} \qquad \widetilde{\varphi}_M(\sigma_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\widetilde{\varphi}_M(\rho_i) : \begin{cases} y_i \longmapsto y_{i+1}^{v_i^{-1}}, \\ y_{i+1} \longmapsto y_i^{v_{i+1}}, \end{cases} \qquad \widetilde{\varphi}_M(\rho_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i \end{cases}$$

is equivalent to the representation φ_M.

Assume that we have a representation $\psi : VB_n \longrightarrow \mathrm{Aut}(H)$ of the virtual braid group into the automorphism group of some group $H = \langle h_1, h_2, \dots, h_m \parallel \mathcal{R} \rangle$, where $\mathcal{R}$ is the set of defining relations.

The following group is assigned to the virtual braid $\beta \in VB_n$:

$$G_\psi(\beta) = \langle h_1, h_2, \dots, h_m \parallel \mathcal{R}, h_i = \psi(\beta)(h_i), \quad i = 1, 2, \dots, m \rangle.$$

The group G_ψ is an invariant of virtual links if the group $G_\psi(\beta)$ is isomorphic to $G_\psi(\beta')$ for each braid β' such that the links $\widehat{\beta}$ and $\widehat{\beta'}$ are equivalent.

This approach is used for the previously defined representation φ_M. Given $\beta \in VB_n$, the group of the braid β is the following group

$$G_M(\beta) = \langle x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n, v_0, v_1, \ldots, v_n || [u_i, u_j] = [v_k, v_l] = [u_i, v_k] = 1,$$

$$x_i = \varphi_M(\beta)(x_i), \quad u_i = \varphi_M(\beta)(u_i), \quad v_i = \varphi_M(\beta)(v_i),$$

$$i, j = 1, 2, \ldots, n, \ k, l = 0, 1, \ldots, n\rangle.$$

Theorem 3 [V. B. – Yu. Mikhalchishina – M. Neshchadim, 2017].

Given $\beta \in VB_n$ and $\beta' \in VB_m$ the two virtual braids such that theirs closures define the same link L, then $G_M(\beta) \cong G_M(\beta')$.

Yu. Mikhalchishina (2017) defined the following three representations of the virtual braid group VB_n into $\mathrm{Aut}(F_{n+1})$, where $F_{n+1} = \langle y, x_1, x_2, \ldots, x_n \rangle$.

1. The representation $W_{1,r}$, $r > 0$ is defined by the action on the generators

$$W_{1,r}(\sigma_i) : \begin{cases} x_i \longmapsto x_i^r x_{i+1} x_i^{-r}, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad W_{1,r}(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{y^{-1}}, \\ x_{i+1} \longmapsto x_i^{y}. \end{cases}$$

2. The representation W_2 is defined by the action on the generators

$$W_2(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1}^{-1} x_i, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad W_2(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{y^{-1}}, \\ x_{i+1} \longmapsto x_i^{y}. \end{cases}$$

3. The representation W_3 is defined by the action on the generators

$$W_3(\sigma_i) : \begin{cases} x_i \longmapsto x_i^2 x_{i+1}, \\ x_{i+1} \longmapsto x_{i+1}^{-1} x_i^{-1} x_{i+1}, \end{cases} \qquad W_3(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{y^{-1}}, \\ x_{i+1} \longmapsto x_i^y. \end{cases}$$

These representations extend Wada representations $w_{1,r}$, $r > 0$, w_2, w_3 of B_n into $\mathrm{Aut}(F_n)$.

Yu. Mikhalchishina for each virtual braid $\beta \in VB_n$ defined three types of groups: $G_{1,r}(\beta)$, $G_2(\beta)$ and $G_3(\beta)$ that correspond to described representations. She proved that these groups are invariants of a virtual link $\widehat{\beta}$.

The Kishino knot is a non-trivial knot that is the connected sum of two trivial knots.

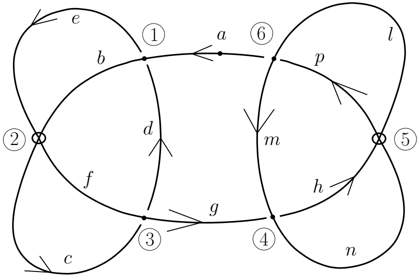

Figure: Kishino knot

Yu. Mikhalchishina proved that groups $G_{1,r}(Ki)$ and $G_2(Ki)$ cannot distinguish the Kishino knot Ki from the trivial one. She formulated the question: whether the group $G_3(Ki)$ is able to distinguish the Kishino knot from the trivial one or not?

Note that the group $G_3(U)$ of the trivial knot U is isomorphic to F_2.

Theorem 3 [V. B. – Yu. Mikhalchishina – M. Neshchadim, ArXiv, 2018].

The group $G = G_3(Ki)$ having generators a, b, c, d and the system of defining relations

$$d^{-1}b^{-d}c^{-2d^{-1}}b^{-d}c^{-2d^{-1}}aa^{-2d}d = a^{-1}b^{-d}c^{-2d^{-1}}a,$$

$$c^{-1}bc = b^{-d}c^{d^{-1}}b^{d},$$

$$c = b^{-d}c^{-2d^{-1}}b^{-d}c^{-2d^{-1}}aa^{-d}a^{-1}c^{2d^{-1}}b^{2d}.$$

is not isomorphic to the free group of rank 2.

Thank you!