~~Group fusion power of association schemes~~

Da Zhao
赵 达

Shanghai Jiao Tong University

jasonzd@sjtu.edu.cn

Aug 7th, 2018, G2R2 @ Novosibirsk

There is a written proof of every mathematical theorem, in Russian.

– I couldn't find the source

- *ОБ ОДНОМ МЕТОДЕ ПОСТРОЕНИЯ ПРИМИТИВНЫХ ГРАФОВ*
  M. Klin 1974
- *О ПРИМИТИВНЫХ КЛЕТОЧНЫХ АЛГЕБРАХ*
  Evdokimov S. A. and Ponomarenko I. N. 1999
- *Primitivity of Permutation Groups, Coherent Algebras and Matrices*
  G. Jones, M. Klin and Y. Moshe 2000

On eigenmatrices of polynomial association schemes

Da Zhao
赵 达

Shanghai Jiao Tong University
Joint work with
Eiichi Bannai & Yaokun Wu
坂 内 英 一 & 吴 耀 琨

jasonzd@sjtu.edu.cn

Aug 7th, 2018, G2R2 @ Novosibirsk

Outline

1. monotone eigenvector of second largest eigenvalue for certain equitable partitions.
2. classification of Q-polynomial association scheme in low dimension.
3. partial dual balanced + Q-polynomial $\implies$ P-polynomial.

Outline

1. **monotone eigenvector of second largest eigenvalue for certain equitable partitions**.
2. classification of Q-polynomial association scheme in low dimension.
3. partial dual balanced + Q-polynomial $\Longrightarrow$ P-polynomial.

Association scheme

Definition

Let X be a finite set of size n. Let $\{R_i\}_{i=0}^{d}$ be a collection of binary relations $R_i \subseteq X \times X$. Let A_i be the corresponding adjacency matrix of R_i.
They satisfy the following properties.

1. $A_0 = I$.
2. $A_0 + A_1 + \cdots + A_d = J$.
3. $A_i^T \in \{A_0, A_1, \ldots, A_d\}$.
4. $A_i A_j = \sum_{k=0}^{d} p_{ij}^{k} A_k$.

Then we call $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ an *association scheme (A.S.)*.

5. $A_i A_j = A_j A_i$. (commutative A.S.)
6. $A_i^T = A_i$. (symmetric A.S. $\implies$ commutative A.S.)

Roughly, a symmetric association scheme is a matrix algebra with 0-1 matrix basis which is closed under transpose, product and entrywise product.

Bose-Mesner algebra

There is another basis for commutative A.S., called idempotents, which are the projection matrices to the common eigenspaces.

1. $E_0 = \frac{1}{|X|}J$.
2. $E_0 + E_1 + \cdots + E_d = I$.
3. $E_i E_j = \delta_{ij} E_i$.
4. $E_i \circ E_j = \sum_{k=0}^{d} q_{ij}^{k} E_k$

1. $A_0 = I$.
2. $A_0 + A_1 + \cdots + A_d = J$.
3. $A_i \circ A_j = \delta_{ij} A_i$.
4. $A_i A_j = \sum_{k=0}^{d} p_{ij}^{k} A_k$

The transition matrices P and Q between the two basis are called eigenmatrices.

$$(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d)P$$
$$|X|\,(E_0, E_1, \ldots, E_d) = (A_0, A_1, \ldots, A_d)Q$$
$$PQ = |X|\, I$$

Polynomial schemes

Definition (metric/P-polynomial)

A symmetric association scheme is called metric (or P-polynomial) if there exists an ordering of relations such that $A_i = v_i(A_1)$, where v_i is a polynomial of degree i.

Definition (cometric/Q-polynomial)

A symmetric association scheme is called cometric (or Q-polynomial) if there exists an ordering of primitive idempotents such that $E_i = v_i^*(E_1)$, where v_i^* is a polynomial of degree i, and the product is Schur product.

- P-polynomial association scheme = distance-regular graph
- Q-polynomial association scheme = ?

Definition (Cosine matrix/Normalized eigenmatrix)

Cosine matrix $C := QM^{-1} = K^{-1}P^{H}$. We index the rows of C by $A_0, A_1, \dots, A_d$ and columns by $E_0, E_1, \dots, E_d$ respectively.

There is a standard result concerning sign-changes of cosines of polynomial schemes.

Theorem

The i-th row (column) of C has exactly i sign-changes, and its difference has exactly $i-1$ sign-changes, if the association scheme is cometric (metric).

Corollary

The 1st row (column) is monotone (decreasing) if the association is cometric (metric).

Block monotonicity of eigenvector

Definition

A matrix B is called blockwise principal unimodal if its columns are blockwise unimodal and the peaks are attained at the diagonal blocks.

$$\left[\begin{array}{c|cc|cc} 8 & 2 & 0 & 0 & 0 \\ \hline 3 & 2 & 3 & 1 & 2 \\ 4 & 3 & 2 & 2 & 1 \\ \hline 0 & 1 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 & 4 \end{array}\right]$$

Theorem (Wu-Z)

Let B be a real block matrix which satisfy the following conditions:

1. $B + cI$ is blockwise principal unimodal for some $c \in \mathbb{R}$,
2. B has constant row sum k,
3. B is diagonalizable.

Let $\rho = \text{MaxReal}\left\{\text{Spec}B - k^1\right\}$, the maximum real eigenvalue of A besides k (if k is a multiple eigenvalue, we only remove one multiplicity). Then there exists a blockwise monotone (right) eigenvector associated to ρ.

Outline

1. monotone eigenvector of second largest eigenvalue for certain equitable partitions.
2. **classification of Q-polynomial association scheme in low dimension**.
3. partial dual balanced + Q-polynomial $\Longrightarrow$ P-polynomial.

Finiteness of metric/cometric schemes

Theorem (Godsil 1988)

There are only finitely many connected co-connected distance-regular graphs with an eigenvalue multiplicity m for all $m \geq 3$.

Theorem (Bang-Dubickas-Koolen-Moulton 2015)

There are only finitely many connected distance-regular graphs of valency k_1 for all $k_1 \geq 3$.

Theorem (Martin-Williford 2009)

There are finitely many cometric association schemes with multiplicity m_1 for all $m_1 \geq 3$.

Theorem (Personal communication with Martin)

There are finitely many cometric association schemes with a relation of valency k for all $k \geq 3$.

Classification of DRGs with small valency

Theorem (Biggs-Boshier-ShaweTaylor 1986)

There are 13 distance-regular graphs of valency $k = 3$.

Theorem (Brouwer-Koolen 1999)

There are 17 possible parameters of distance-regular graphs of valency $k = 4$, each of which is determined and unique except possibly one parameter.

We aim to finish the classification dual to these two theorems.

Spherical representation

Definition (spherical representation)

The spherical representation of a symmetric A.S. $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ with respect to E_i is the mapping $X \to \mathbb{R}^{m_i}$ defined by

$$x \to \overline{x} = \sqrt{\frac{|X|}{m_i}} E_i \phi_x$$

where ϕ_x is the characteristic vector of x.

The image is on the unit sphere $S^{m_i - 1} \subset \mathbb{R}^{m_i}$.
We identify X and $\overline{X}$ when the embedding is faithful.

Classification of dual DRGs in $\mathbb{R}^3$

Theorem (Bannai-Bannai 2006)

The only primitive association scheme with $m_1 = 3$ is tetrahedron.

Theorem (Bannai-Z)

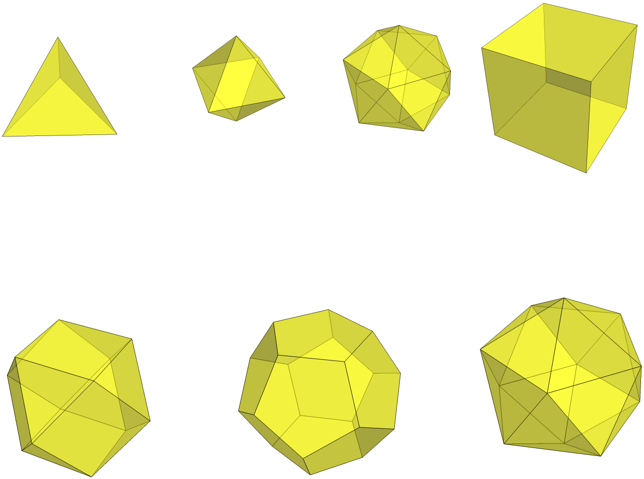
Let $\mathfrak{X}$ be a symmetric association scheme. If $\mathfrak{X}$ has a faithful spherical embeddings X with $m_1 = 3$ in $\mathbb{R}^3$, then it must be one of the followings:

1. the regular tetrahedron $(|X| = 4)$;
2. the regular octahedron $(|X| = 6)$;
3. the cube $(|X| = 8)$;
4. the regular icosahedron $(|X| = 12)$;
5. the quasi-regular polyhedron of type $[3, 4, 3, 4]$ $(|X| = 12)$;
6. the regular dodecahedron $(|X| = 20)$;

*The quasi-regular polyhedron of type $[3, 5, 3, 5]$ $(|X| = 30)$ is non-commutative.

Corollary

The Q-polynomial association schemes with $m_1 = 3$ are (1-4) in the above list.



Partial classification of dual DRGs in $\mathbb{R}^4$

Theorem (Bannai-Z)

Let $\mathfrak{X}$ be a Q-polynomial association scheme with $m_1 = 4$ in $\mathbb{R}^4$, then one of the following holds.

1. $k_1 = 3$, $\mathfrak{X}$ is Petersen graph O_3, or complete bipartite graph $K_{3,3}$;
2. $k_1 = 4$, $\mathfrak{X}$ is complete graph K_5, 5-cross polytope, $L(K_{3,3})$, or 4-cube Q_4;
3. $k_1 = 5$, $\mathfrak{X}$ is geometrically locally pentagon;
4. $k_1 = 6$, $\mathfrak{X}$ is geometrically locally (twisted) 3-prism; (16-cell is an example)
5. $k_1 = 7$, impossible;
6. $8 \le k_1 \le 12$, ongoing. (24-cell is an example).

Outline

1. monotone eigenvector of second largest eigenvalue for certain equitable partitions.
2. classification of Q-polynomial association scheme in low dimension.
3. **partial dual balanced + Q-polynomial $\Longrightarrow$ P-polynomial**.

Conjectures revisited

Conjecture (Babai)

The maximum valency of a primitive association scheme is bounded by a function of the minimum (non-trivial) valency, i.e., $k_{\max} \leq f(k_{\min})$.

Conjecture (Bannai-Ito)

{primitive metric A.S.} = {primitive cometric A.S.} for large class number d.

Distribution graph and Representation graph

Definition (distribution graph)

Given an association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$, the distribution graph Δ_{A_i} with respect to A_i is a graph whose vertices are $0, 1, \ldots, d$, and two vertices j and k are adjacent if and only if $p_{ij}^{k} > 0$.

Definition (representation graph)

Given an association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$, the representation graph Δ_{E_i} with respect to E_i is a graph whose vertices are $0, 1, \ldots, d$, and two vertices j and k are adjacent if and only if $q_{ij}^{k} > 0$.

$\mathfrak{X}$ is P-polynomial w.r.t A if and only if Δ_A is a path.
$\mathfrak{X}$ is Q-polynomial w.r.t E if and only if Δ_E is a path.

Balanced condition

Definition (balanced)

Let $x, y \in X$ and $i, j \in \{0, 1, \ldots, d\}$ be arbitrarily fixed. Then we say the spherical representation $\rho = \rho_E$ is balanced, if there exists an $\alpha \in \mathbb{R}$, such that

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} \rho(z) - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} \rho(z) = \alpha(\rho(x) - \rho(y)). \quad (1)$$

For $(x, y) \in R_k$, we have $\alpha = \gamma_{ij}^k$, where

$$\gamma_{ij}^k := p_{ij}^k \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_k^*}.$$

If 1 holds for fixed i and j, then ρ is called $\{i, j\}$-**balanced**.

Theorem (Terwilliger 1987)

Suppose $\mathfrak{X}$ is Q-polynomial with respect to E, then the spherical embedding ρ_E is balanced.

Relation between metric and cometric schemes

Theorem (Terwilliger 1987)

Let $\mathfrak{X}$ be a P-polynomial association scheme. Suppose one of its representation graph is a tree, then it is a path, hence $\mathfrak{X}$ is Q-polynomial.

Theorem (Bannai-Bannai-Ito Book)

Let $\mathfrak{X}$ be a P-polynomial association scheme. Suppose $\rho = \rho_E$ is a non-degenerate spherical representation and it is $\{1, 2\}$-balanced, then $\mathfrak{X}$ is Q-polynomial with respect to E.

Theorem

Let $\mathfrak{X}$ be a Q-polynomial association scheme. Suppose A is a connecting relation and it is $\{1, 2\}$-dual balanced, then $\mathfrak{X}$ is P-polynomial with respect to A.

Scaffold – tensor represented by diagrams

Definition (Scaffold)

Let X be an index set (for basis), and let $\mathbb{A}$ be a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$.
Given a digraph $G = (V(G), E(G))$.
Take a subset $R \subseteq V(G)$ be the red nodes.
Assign matrix weights to arcs $w : E(G) \to \mathbb{A}$.
The scaffold $S(G, R, w)$ is defined as follows.

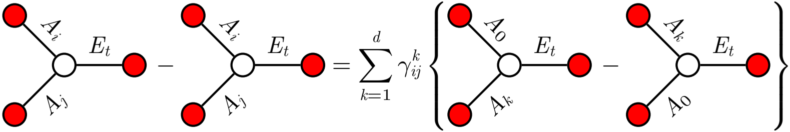
$$S(G, R, w) = \sum_{\varphi : V(G) \to X} \prod_{e=(a,b) \in E(G)} w(e)_{\varphi(a),\varphi(b)} \bigotimes_{r \in R} \widehat{\varphi(r)}$$

It is an element of $V^{\otimes |R|}$. The white nodes are the ‘dumb variables’.

Balanced condition rewritten

Definition

A primitive idempotent E_t is called balanced if there exists γ_{ij}^k such that the following holds for all i, j.



(2)

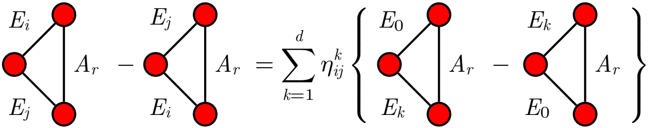
If Equation (2) holds for fixed i and j, then E_t is called $\{i, j\}$-balanced.

If Equation (2) holds, then $\gamma_{ij}^k = p_{ij}^k \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_k^*}$, where θ_i^* is the eigenvalue of A_i associated to E_t.

Dual balanced condition

Definition

An adjacency relation A_r is called dual balanced if there exists η_{ij}^k such that the following holds for all i, j.

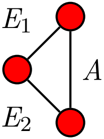


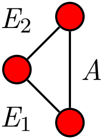
(3)

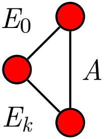
If Equation (3) holds for fixed i and j, then A_r is called $\{i, j\}$-dual balanced.

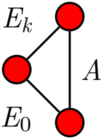
If Equation (3) holds, then $\eta_{ij}^k = q_{ij}^k \frac{\theta_i - \theta_j}{\theta_0 - \theta_k}$, where θ_i is the eigenvalue of A_r associated to E_i.

Proofs get simplified

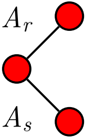
$$0 =$$ 

$$-$$ 

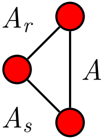
$$- \sum_{k=1}^{3} \eta_{12}^{k} \Big\{$$ 

$$-$$ 

$$\Big\}.$$

Taking entrywise product of both sides with 

, we obtain

$$0 = \left\{ \theta_r^* v_2^*(\theta_s^*) - \theta_s^* v_2^*(\theta_r^*) - \sum_{k=1}^{3} \eta_{12}^{k} [v_k^*(\theta_s^*) - v_k^*(\theta_r^*)] \right\}$$ 

$$.$$

Спасибо за внимание!