Minimum supports of eigenfunctions in bilinear forms graphs

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Checklist: Minimum support problem

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- Eigenfunctions: adjacency matrix/local definition

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Example: $n = m = 2$, F_2

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

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Local structure

$(q - 1)$-clique extension of $\begin{bmatrix} n \\ 1 \end{bmatrix}_q \times \begin{bmatrix} m \\ 1 \end{bmatrix}_q$-lattice

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Canonical directions

Let $\{e_i \mid e_i \in F_q^m\}$ be a set of vectors with a first non-zero element equal to 1. Example: $[0,1]$, $[1,0]$, $[1,1]$, $[1,2]$, $[1,3]$, $[1,4]$ (case F_5)

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Equivalence classes in $F^*_{q^n}$

Let $\{\delta_i \mid \delta_i \in F^*_{q^n}\}$ be a set of column-vectors with a first non-zero element equal to 1.

Example: $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\a_1\end{bmatrix}, \begin{bmatrix}1\\a_2\end{bmatrix}, \ldots, \begin{bmatrix}1\\a_{q-1}\end{bmatrix}$ (case F_{q^2})

Denote $K(\delta_i) = \{a_t \cdot \delta_i \mid a_t \in F^*_q\}$

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Neighbours of U

$$U + a_t \cdot \delta_j \cdot e_i$$

Strongly-regular case: $Bil_q(n, 2)$

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Case $Bil_p(2,2)$ where p is prime:

Theorem: *Let* a_1 *be a generating element of the multiplicative group* F_p^*. *Denote* $a_0 = 0$; $a_2 = a_1^2$; $\dots$; $a_{p-2} = a_1^{p-2}$; $a_{p-1} = a_1^{p-1} = 1$. *Choose* $\delta \in F_p$, *such that* $\delta \neq -\xi^2$ *for all* $\xi \in F_p$. *The independent set*

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{a_i^2\delta+1} & \frac{a_i}{a_i^2\delta+1} \\ \frac{a_i\delta}{a_i^2\delta+1} & \frac{a_i^2\delta}{a_i^2\delta+1} \end{bmatrix}$$

together with the vertices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{a_i^2\delta+1} & \frac{-a_i}{a_i^2\delta+1} \\ \frac{a_i\delta}{a_i^2\delta+1} & \frac{1}{a_i^2\delta+1} \end{bmatrix}$$

, where $i = 0 \dots p-1$, *form a minimum eigensupport as two parts of a complete bipartite graph* $K_{p+1,p+1}$.

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Spoiler alert: No

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Why do we need it?

$Bil_q(n,m)$ can be considered as a subgraph of $J_q(n+m,m)$ as follows: given a fixed subspace W of a dimension n, all m-spaces U such that $U \cap W = \emptyset$ are the vertices of $Bil_q(n,m)$

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- Contradiction.

Thank you for your attention!