Classification of t-balanced regular Cayley maps on some groups

Young Soo Kwon
(Yeungnam University, Korea)
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Outline

1. Introduction to maps, regular maps, Cayley maps and regular Cayley maps

2. Skew-morphisms and their properties

3. Some known results

4. Classification of t-balanced regular Cayley maps on some groups

5. Future research

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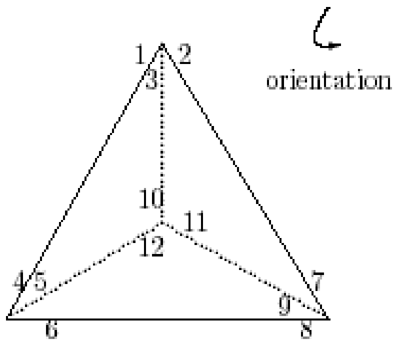

$D = \{1,2,3,\ldots,12\}$

$R = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$

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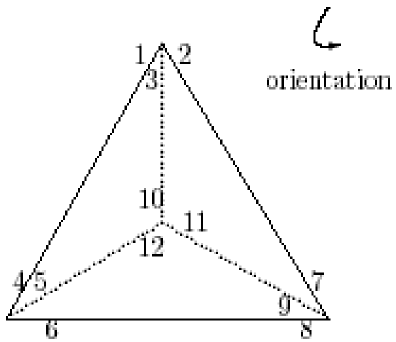

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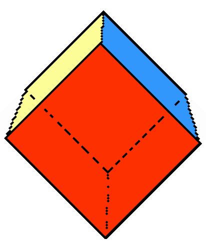

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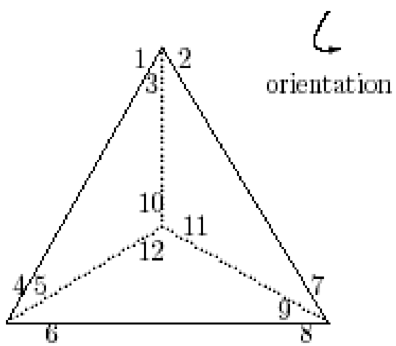

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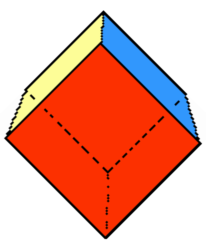
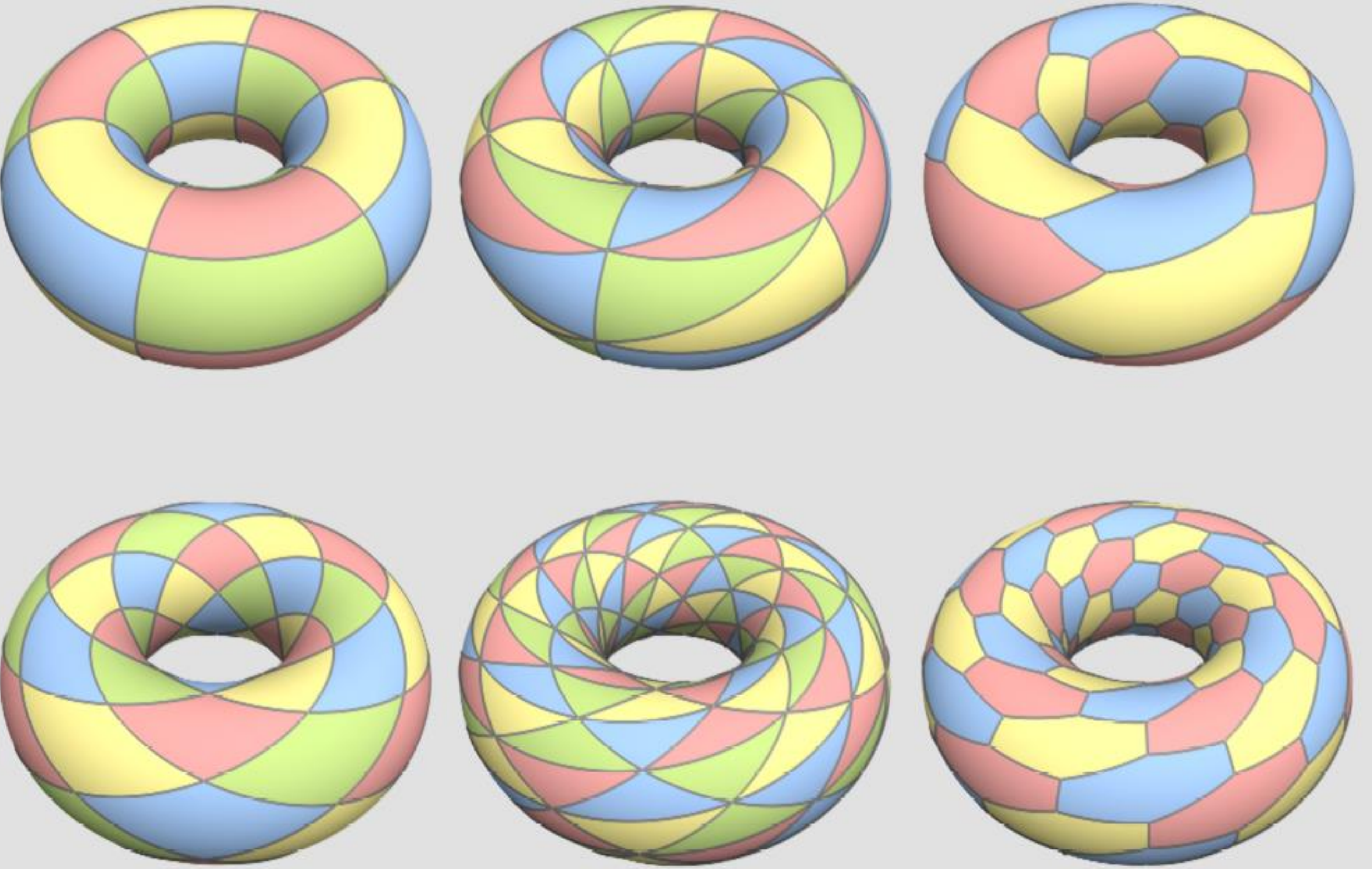


A *map automorphism*: graph auto. extended to a surface homeo.

$\mathrm{Aut}^+(\mathfrak{M})=S_4$ acts regularly on arc(incident vertex-edge pair) set.

Regular map



[Definition]

1. For a group Γ and a set $X \subset \Gamma$ such that $X^{-1} = X$, a Cayley graph Cay($\Gamma : X$)=(V, E) is a graph such that $V = \Gamma$ and $E = \{\{g, gx\} \mid x \in X\}$.

2. For any $g \in \Gamma$, let $L_g : \Gamma \to \Gamma$ such that $L_g(h) = gh$ for any $h \in \Gamma$. Let $L_\Gamma = \{L_g \mid g \in \Gamma\}$.

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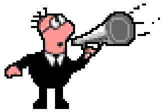
Example:

$$G = Cay(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 : \{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}) \Rightarrow \mathrm{G} = \mathrm{C}_5 \square \mathrm{C}_5 \square \mathrm{C}_5$$



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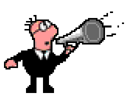


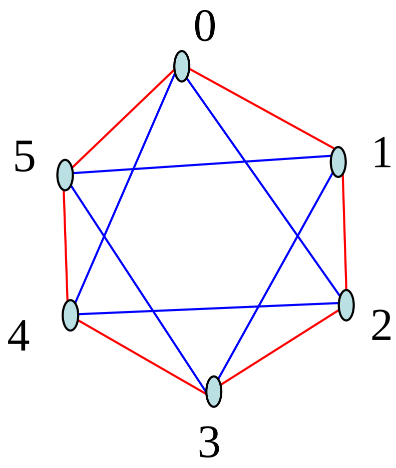
[Definition]

$p(x)^{-1} = p^t(x^{-1}) \Rightarrow$ t-balanced Cayley map.

$t = 1 \Rightarrow$ balanced $\qquad t = -1 \Rightarrow$ antibalanced

Example:

$$G = Cay(\mathbb{Z}_6 : \{1,2,4,5\}, p=(1,5,4,2))$$



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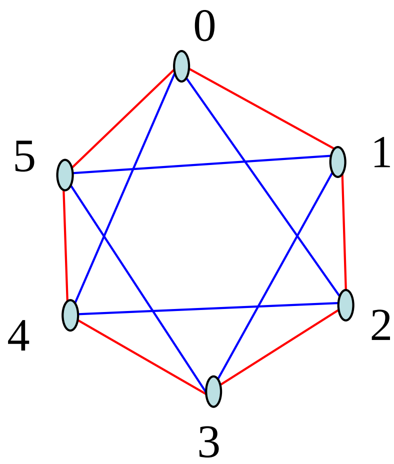

$triangle : 2$

$hexagon : 1$

$12 - gon : 1$

$v - e + f = 6 - 12 + 4 = -2$

supporting surface: double torus

Skew-morphisms and their properties

For a group Γ, a bijection $\phi:\Gamma\to\Gamma$ is called skew-morphism with power function $\pi:\Gamma\to\mathbb{Z}$ if $\phi(1_\Gamma)=1_\Gamma$ and $\phi(gh)=\phi(g)\phi^{\pi(g)}(h)$ for all $g,h\in\Gamma$.

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(complementary product of G and A)

$$\Rightarrow\ \Gamma=\{ga^i\mid g\in G,\ a^i\in A\}=\{a^ig\mid g\in G,\ a^i\in A\}$$

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$\Rightarrow\ \phi(gh)=\phi(g)\phi^{\pi(g)}(h)$, namely, ϕ is a skew-morphism of Γ.

Conversely, let ϕ be a skew-morphism of G w.r.t. a power fuction π.

$G\langle\phi\rangle$ is a subgroup of Sym(G).

[Lemma]

ϕ: a skew-morphism of a group G w.r.t a power function π $\Rightarrow$

1. $Ker(\phi)=\{g\in G \mid \pi(g)=1\}\leq G$.

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5. For any automorphism γ of G, $\gamma^{-1}\phi\gamma$ is also a skew-morphism of G.

[Lemma]

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A skew-morphism of Γ containing an orbit O satisfying $O^{-1} = O$ and $\Gamma = \langle O \rangle$: Cayley skew.

other skew-morphisms: nonCayley.

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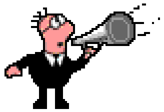

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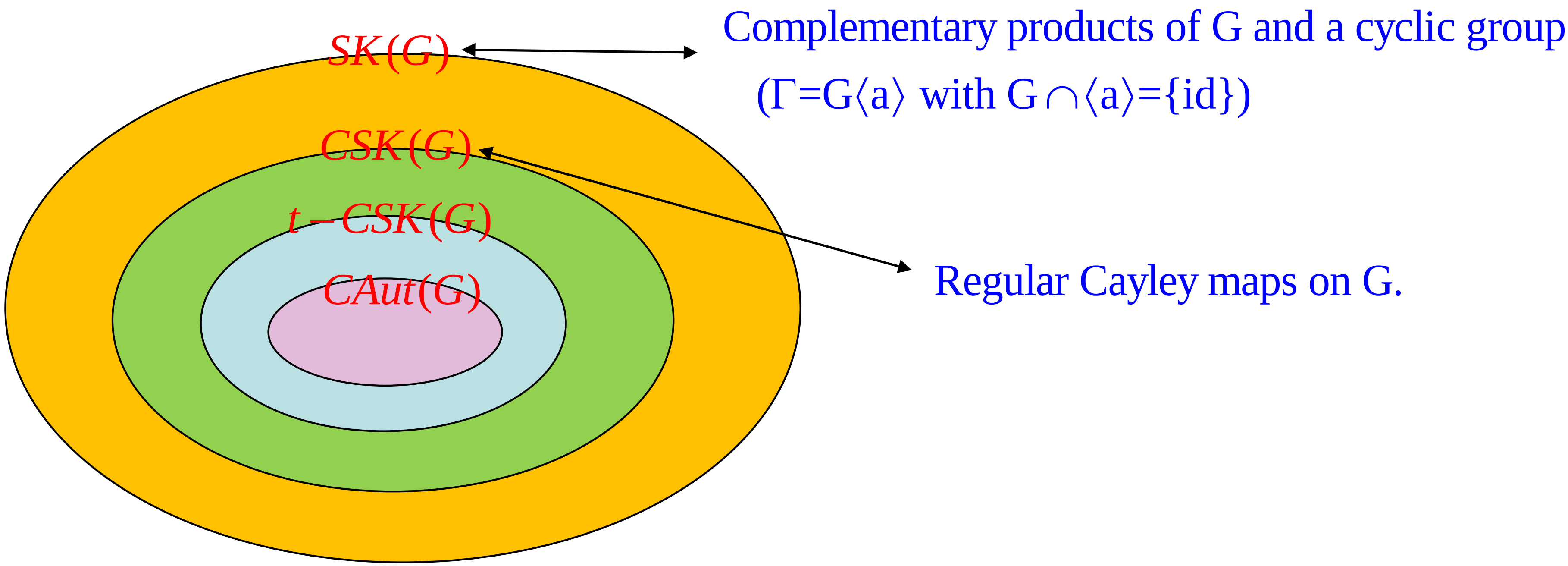
Note that if ϕ is a Cayley skew. and there is a corresponding t-balanced Cayley map , we also call ϕ t-balanced skew-morphism.


SK(G)
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t−CSK(G)
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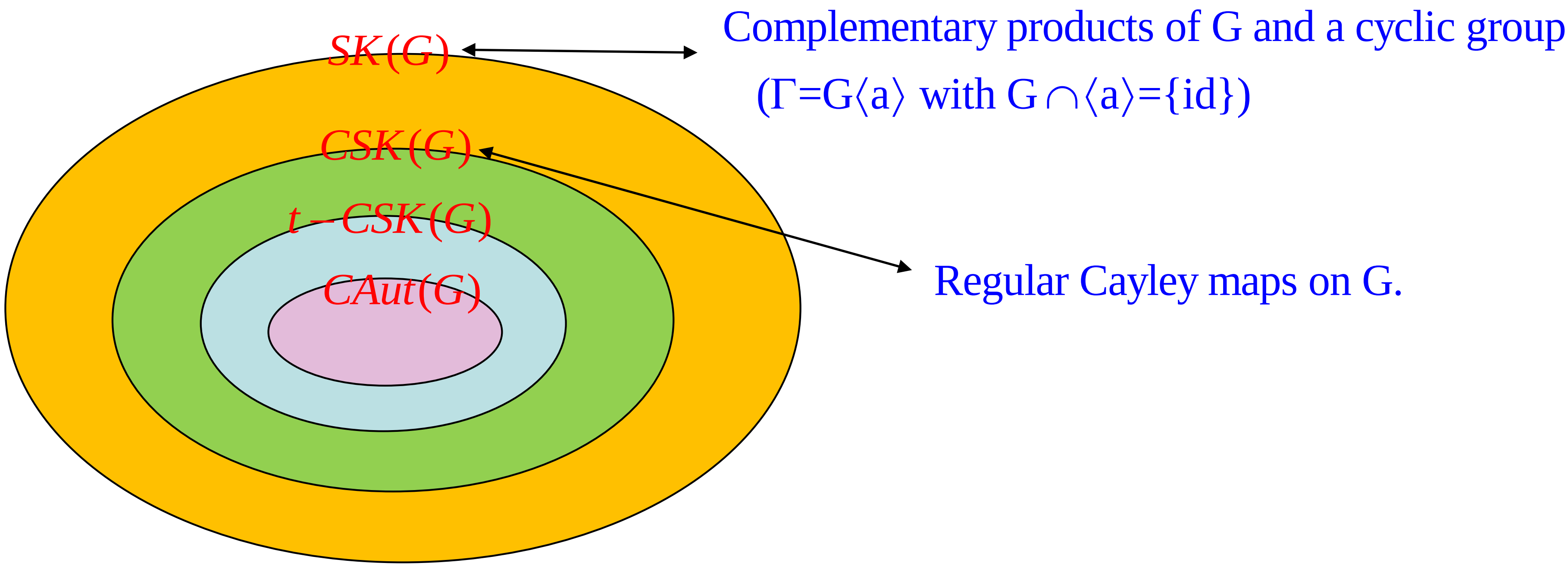
Complementary products of G and a cyclic group.
(Γ=G⟨a⟩ with G∩⟨a⟩={id})
Regular Cayley maps on G.



[Some Results]

1. Cyclic groups for CSK. ('11 TAMS, M. Conder and T. Tucker)
2. Dihedral groups for SK and CSK. (18+ Hu, Kovacs, K)
3. Finite simple groups for CSK. (17 M. Conder et al.)

[Open Problems]

Classification of all skew-morphisms of cyclic groups

[Lemma]

1. A balanced Cayley map $CM(\Gamma : X, p)$ is regular $\Leftrightarrow$ there exists a group automorphism ϕ of Γ such that $\phi(X) = X$ and $\phi|_X = p$.

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(1) Im(π)={1,t} (2) Ker(π)=π^{-1}(1)=Γ^{+}

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Γ^{+} : even-word subgroup of Γ ([$\Gamma : \Gamma^{+}$] = 1 or 2)

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$$\phi(gh) = \phi(g)\phi^{\pi(g)}(h) = \begin{cases} \phi(g)\phi(h) \text{ if } g \in \Gamma^+ \\ \phi(g)\phi^t(h) \text{ if } g \in \Gamma - \Gamma^+ \end{cases}.$$

Some Known results

1. *A*nti-balanced regular Cayley maps on abelian groups

('07 JCTB M. Conder, R. Jajcay and T. Tucker)

(1) $\Gamma = \mathbb{Z}_{2n}$, $\phi(2k) = 2ks$, $\phi(2k-1) = 2ks+1$, where $s^2 \equiv 1 \pmod{n}$

(2) $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_2$, $\phi(k,0) = (k,0)$, $\phi(k,1) = (k+1,1)$

(3) $\Gamma = \mathbb{Z}_{2mn} \times \mathbb{Z}_m$, $\phi(2k, j) = (2k, k-j)$, $\phi(2k+1, j) = (2k-1, k-j)$.

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2. *t*-balanced regular Cayley maps on dihedral groups

('06 EJC, J.H. Kwak, K, R. Feng)

(1) *balanced*

$\phi(a^j) = a^{js},\ \phi(a^j b) = a^{js+1}b.$

$D_n = \langle a, b \mid a^n = b^2 = abab = 1 \rangle$

(2) $t-balanced\,(t>1)$

(i) n:even, $(2us+1)^m \equiv -1 \pmod{n}$, (ii) $2s^2 \equiv 2(2us+1) \pmod{n}$

$\phi(a^{2j}) = a^{2js},\ \phi(a^{2j+1}) = a^{2js+2u},\ \phi(a^{2j}b) = a^{2js+1},\ \phi(a^{2j+1}b) = a^{2js+2u+1}$

$(2m+1)-balanced.$

4. t-balanced regular Cayley maps on cyclic groups ('10 DM, K)

(1) balanced regular Cayley map

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(2) $t-balanced\,(t>1)$

n: even, $\phi(2j) = 2js$, $\phi(2j+1) = 2js + 2u + 1$ s. t.

$\exists\, n_1$ and n_2 satisfying

(i) $n = n_1 n_2$, $(n_1, n_2) = 1$ (ii) $(s, \frac{n}{2}) = 1$

(iii) $s \equiv u \equiv 1 \pmod{n_1}$ (iv) $2u+1 \equiv s \pmod{n_2}$

(v) $\exists\, m$ s. t. $2u(1 + s + \cdots + s^{m-1}) \equiv -2 \pmod{n}$

$\phi \pmod{n_1}$: antibalanced

$\phi \pmod{n_2}$: balanced

Classification of t-balanced regular Cayley maps on some groups

$$\Gamma(n,r)=\langle a,b \mid a^n=b^2=1, bab=a^r\rangle,\ r^2\equiv 1(\bmod n)$$

$$r=-1 \quad\Rightarrow\quad \Gamma(n,r)\simeq D_n$$

$$r=1 \quad\Rightarrow\quad \Gamma(n,r)\simeq \mathbb{Z}_n\times\mathbb{Z}_2$$

$$r=\frac{n}{2}-1 \quad\Rightarrow\quad \Gamma(n,r)\simeq SD_n$$

$\langle a\rangle<\Gamma(n,r).$ A-type

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Let $n=2^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}=2^{\alpha}n'$

Note that $r\equiv\pm1 \pmod{p_i^{\alpha_i}}$ and $r\equiv\pm1,\ 2^{\alpha-1}\pm1 \pmod{2^{\alpha}}$.

Classification of t-balanced regular Cayley maps on some groups

$$\Gamma(n,r) = \langle a,b \mid a^n = b^2 = 1, bab = a^r \rangle,\ r^2 \equiv 1 (\mathrm{mod}\, n)$$

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$$r = 1 \quad \Rightarrow \quad \Gamma(n,r) \simeq \mathbb{Z}_n \times \mathbb{Z}_2$$

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Let $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = 2^{\alpha} n'$

Note that $r \equiv \pm 1 \ (\mathrm{mod}\ p_i^{\alpha_i})$ and $r \equiv \pm 1,\ 2^{\alpha-1} \pm 1 \ (\mathrm{mod}\ 2^{\alpha})$.

[Lemma] ϕ: t-balanced $(n \geq 5) \Rightarrow \phi|_{\langle a^2 \rangle}$: auto. of $\langle a^2 \rangle \Rightarrow$

1. $\phi \ (\mathrm{mod}\ 2p_i^{\alpha_i})$: t_i-balanced and $\phi \ (\mathrm{mod}\ 2^{\alpha})$: t'-balanced

2. BB-type or AB-type.

ϕ: a Cayley skew morphism corresponding to a t-balanced. $\Rightarrow$

(1) $r \equiv 1 \pmod{p_i^{\alpha_i}}$ $\Rightarrow$

$\phi(a^j b^k) = a^{js} b^k$ with $s^m \equiv -1 \pmod{p_i^{\alpha_i}}$ BB-type, balanced

$\phi(a^j) = a^j$, $\phi(a^j b) = a^{j+1} b$ BB-type, antibalanced

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(2) $r \equiv -1 \pmod{p_i^{\alpha_i}} \Rightarrow$

$\phi(a^j) = a^{js},\ \phi(a^j b) = a^{js+1} b$ BB-type, balanced

$\phi(a^{2j}) = a^{2js},\ \phi(a^{2j+1}) = a^{2js+2u} b,\ \phi(a^{2j} b) = a^{2js+1},\ \phi(a^{2j+1} b) = a^{2js+2u+1} b$

(i) $s^{2m} \equiv -1 \pmod{2p_i^{\alpha_i}}$, (ii) $s^2 \equiv 2us + 1 \pmod{2p_i^{\alpha_i}}$

AB-type, (2m+1)-balanced

(3) $r \equiv 1 \pmod{2^\alpha} \Rightarrow$

(i) $\alpha = 1 \Rightarrow \phi = (a\ ab\ \ b)$ or $\phi = (a\ b)$ or $\phi = (a\ ab)$ or $(b\ \ ab)$

(ii) $\alpha = 2,\ \phi = (a\ ab\ \ a^3\ \ a^3 b)(b\ \ a^2 \mathrm{b})$ AB-type, balanced

(iii) $\alpha \geq 2 \Rightarrow \phi(a^j) = a^j,\ \phi(a^j b) = a^{j+1} b$ BB-type, antibalanced.

$$\phi(a^{2j}) = a^{2j},\ \ \phi(a^{2j+1}) = a^{2j+2} b,\ \ \phi(a^{2j} b) = a^{2j+1},\ \ \phi(a^{2j+1} b) = a^{2j+1} b$$

AB-type, antibalanced

(iv) $\alpha \geq 3 \Rightarrow$

$\phi(a^j) = a^{j(1+2^{\alpha-1})},\ \ \phi(a^j b) = a^{j(1+2^{\alpha-1})+1} b$ BB-type, $(2^{\alpha-1}$-1)-balanced.

$$\phi(a^{2j}) = a^{2j},\ \ \phi(a^{2j+1}) = a^{2j+2^{\alpha-1}+2} b,\ \ \phi(a^{2j} b) = a^{2j+1},\ \ \phi(a^{2j+1} b) = a^{2j+1} b$$

AB-type, $(2^{\alpha-1}$-1)-balanced

(3) $r \equiv 1 \pmod{2^\alpha} \Rightarrow$

(i) $\alpha=1 \Rightarrow \phi=(a\ ab\ \ b)$ or $\phi=(a\ b)$ or $\phi=(a\ ab)$ or $(b\ \ ab)$

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AB-type, $(2^{\alpha-1}$-1)-balanced

(4) $r \equiv -1 \pmod{2^\alpha} \Rightarrow$

$\phi(a^j)=a^{js}$, $\phi(a^jb)=a^{js+1}b$ BB-type, balanced

α=2 and $\phi=(b\ \ a\ \ a^2b\ \ a^{-1})(ab\ \ a^{-1}b)$ AB-type, antibalanced

α=2 and $\phi=(b\ \ a\ \ a^{-1})(a^2\ \ a^3b\ ab)$ antibalanced

(5) $r \equiv 2^{\alpha-1}+1 \pmod{2^{\alpha}} \Rightarrow$

$\phi(a^j)=a^j,\ \phi(a^j b)=a^{j+1}b$ BB-type, ($2^{\alpha-1}$-1)-balanced

$\phi(a^j)=a^{j(1+2^{\alpha-1})},\ \phi(a^j b)=a^{j(1+2^{\alpha-1})+1}b$ BB-type, antibalanced

$\phi(a^{2j})=a^{2j},\ \phi(a^{2j+1})=a^{2j+2}b,\ \phi(a^{2j}b)=a^{2j+1},\ \phi(a^{2j+1}b)=a^{2j+1}b$

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AB-type, antibalanced

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AB-type, ($2^{\alpha-1}$-1)-balanced

(6) $r \equiv 2^{\alpha-1}-1 \pmod{2^{\alpha}} \Rightarrow$

$\phi(a^j)=a^{js},\ \phi(a^j b)=a^{js+1}b$ with $s^{2m-1}+s^{2m-2}+\cdots+1 \equiv 2^{\alpha-1} \pmod{2^{\alpha}}$

BB-type, (2m+1)-balanced

[proof: $n = 2^{\alpha}$, $r = 1$ ($\Gamma = \mathbb{Z}_{2^{\alpha}} \times \mathbb{Z}_2$)]

$\alpha = 1 \Rightarrow$ $\phi = (a\ ab\ \ b)$ or $\phi = (a\ b)$ or $\phi = (a\ ab)$ or $(b\ \ ab)$

$\alpha = 2 \Rightarrow$ $\phi = (a\ ab\ \ a^3\ \ a^3 b)(b\ \ a^2 \mathrm{b})$ AB-type, balanced

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balanced $\Rightarrow$ ϕ : auto. of Γ $\Rightarrow$ $\phi(a) = a^j$ or $a^j b$ with odd j, $\phi(b) = b$ or $a^{\alpha-1}\mathrm{b}$

$\Rightarrow$ In any cases, $\exists$ no generating orbit which is closed under inverse.

$\Rightarrow$ $\exists$ no balanced Cayley map.

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t-balanced, Ker(ϕ)=$\langle a\rangle$ $\Rightarrow$ $\phi(a)=a^s,\ \phi(b)=a^kb\ \ (n,s)=1,\ k$:odd $\Rightarrow$

assume $k=1$ $\Rightarrow$ $a^{s+1}b=\phi(ab)=\phi(ba)=\phi(b)\phi^t(a)=aba^{s^t}$ $\Rightarrow$ $s^{t-1}=1$

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generating orbit: $(a^jb\ \ a^{js+1}b\ \ a^{js^2+s+1}b\cdots)$ $\Rightarrow$

$\exists\ k$ s. t. $js^k+s^{k-1}+\cdots+1=-j$ $\Rightarrow$ $js^{k+t}+s^{k+t-1}+\cdots+1=-js-1$

$$-js-1 = js^{k+t} + s^{k+t-1} + \cdots + 1 = s^{t}(js^{k} + s^{k-1} + \cdots + 1) + s^{t-1} + \cdots + 1$$

$$= s(-j) + s^{t-1} + \cdots + 1$$

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$$s^{t-1} + \cdots + 1 = -1 \Rightarrow (s-1)(s^{t-1} + \cdots + 1) = -(s-1) \Rightarrow s^{t} - 1 = -(s-1)$$

$2(s-1) = 0 \Rightarrow s = 1$ or $s = 2^{\alpha-1} + 1$

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$2(s-1) = 0 \Rightarrow s = 1$ or $s = 2^{\alpha-1} + 1$

$s = 1 \Rightarrow \phi(a^j) = a^j,\ \phi(a^j b) = a^{j+1} b$ BB-type, antibalanced

$s = 2^{\alpha-1} + 1 \Rightarrow \phi(a^j) = a^{j(1+2^{\alpha-1})},\ \phi(a^j b) = a^{j(1+2^{\alpha-1})+1} b$ BB-type, $(2^{\alpha-1}$-1)-balanced

$$-js-1 = js^{k+t} + s^{k+t-1} + \cdots + 1 = s^t(js^k + s^{k-1} + \cdots + 1) + s^{t-1} + \cdots + 1$$
$$= s(\text{-}j) + s^{t-1} + \cdots + 1$$

$$s^{t-1} + \cdots + 1 = -1 \Rightarrow (s-1)(s^{t-1} + \cdots + 1) = -(s-1) \Rightarrow s^t - 1 = -(s-1)$$

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$s = 2^{\alpha-1} + 1 \Rightarrow \phi(a^j) = a^{j(1+2^{\alpha-1})},\ \phi(a^j b) = a^{j(1+2^{\alpha-1})+1} b$ BB-type, $(2^{\alpha-1}$-1)-balanced

t-balanced, Ker(ϕ)=$\langle a^2, b\rangle$ $\Rightarrow$ $\phi(a^2) = a^{2s},\ \phi(b) = b$ or $a^{\alpha-1} b$

generating orbit: $(a \ \ a^{2k+1}b \ \ a^{2k(s+1)+1} \cdots)$ or $(a \ \ a^{2k+1}b \ \ a^{2k(s+1)+2^{\alpha-1}+1} \cdots) \Rightarrow$
all exponents of A-type elements are 1 modulo 4 $\Rightarrow$ not inverse closed

$\Rightarrow$ $\exists$ no t-balanced Cayley map with Ker$(\phi) = \langle a^2, b\rangle$.

t-balanced, Ker(ϕ)=$\langle a^2, ab\rangle$ $\Rightarrow$

$$\phi(a^{2j}) = a^{2j},\ \ \phi(a^{2j+1}) = a^{2j+2}b,\ \ \phi(a^{2j}b) = a^{2j+1},\ \ \phi(a^{2j+1}b) = a^{2j+1}b$$

AB-type, antibalanced

$$\phi(a^{2j}) = a^{2j},\ \ \phi(a^{2j+1}) = a^{2j+2^{\alpha-1}+2}b,\ \ \phi(a^{2j}b) = a^{2j+1},\ \ \phi(a^{2j+1}b) = a^{2j+1}b$$

AB-type, ($2^{\alpha-1}$-1)-balanced

[Lemma]

For any positive integers $i_1,\ i_2,\ n_1,\ n_2,$ there exists a positive integer a s. t. $a \equiv i_1 \pmod{n_1}$ and $a \equiv i_2 \pmod{n_2} \Leftrightarrow (n_1,\ n_2) \mid |i_2 - i_1|$.

For a positive integer s with $(n,s)=1$, let $o(s,n)$ be the smallest positive integer m s. t. $1+s+s^2+\cdots+s^{m-1} \equiv 0 \pmod{n}$.

Let $n^+ = \prod_{r\equiv 1 \pmod{p_i^{\alpha_i}}} p_i^{\alpha_i}$, $n^- = \prod_{r\equiv -1 \pmod{p_i^{\alpha_i}}} p_i^{\alpha_i}$ and $n' = n^+ n^-$.

[BB-type, balanced]

$r \equiv -1 \pmod{2^\alpha}$

ϕ_1: $\phi(a^j)=a^{js},\ \phi(a^j b)=a^{js+u}b \quad (n,s)=1 \quad u \equiv 0 \pmod{n^+},\ u \equiv 1 \pmod{n^-}$

$\exists\, m$ s. t. $s^m \equiv -1 \pmod{n^+}$, $(2m, o(s,2^\alpha n^-)) = (m, o(s,2^\alpha n^-))$

[BB-type, t-balanced $(t>1)$]

Case 1. $\phi(a^j)=a^{js}$, $\phi(a^jb)=a^{js+u}b$ $\quad (n,s)=1$

$u\equiv 0 \pmod{n^+}$, $u\equiv 1 \pmod{n^-}$

$\exists\, m$ s. t. $s^m\equiv -1 \pmod{n^+}$, $(2m, o(s,n^-)) = (m, o(s,n^-))$

$r\equiv 1$ or $2^{\alpha-1}+1 \pmod{2^\alpha}$

$\phi_2 : s\equiv u\equiv 1 \pmod{2^\alpha}$ $\quad \alpha\geq 2 \Rightarrow o(s,n')\equiv 4 \pmod 8$

$r\equiv 1$ or $2^{\alpha-1}+1 \pmod{2^\alpha}$ with $\alpha\geq 3$

$\phi_3 :$ $s\equiv 1+2^{\alpha-1}$, $u\equiv 1 \pmod{2^\alpha}$ $\quad o(s,n')\equiv 4 \pmod 8$

$r\equiv 2^{\alpha-1}$-1 $\pmod{2^\alpha}$ with $\alpha\geq 3$

$\phi_4 :$ $u\equiv 1 \pmod{2^\alpha}$ $\quad \exists\, 2m_1$ s. t.

$s^{2m_1}\equiv -1 \pmod{n^+}$ and $1+s+\cdots+s^{2m_1-1}\equiv 2^{\alpha-1}n^- \pmod{2^\alpha n^-}$

Case 2. $\phi(a^j)=a^{js}, \ \phi(a^j b)=a^{js+u}b \quad (n,s)=1$

$\mathrm{s} \equiv u \equiv 1 \ (\mathrm{mod\ n}^+), \ u \equiv 1 \ (\mathrm{mod\ n}^-) \qquad (n^+, o(s, n^-)) = 1$

$r \equiv 1 \text{ or } 2^{\alpha-1}+1 \ (\mathrm{mod}\ 2^\alpha)$

$\phi_5 : s \equiv u \equiv 1 \ (\mathrm{mod}\ 2^\alpha) \quad (2^\alpha n^+, o(s, n^-)) = 1 \text{ or } 2$

$r \equiv 1 \text{ or } 2^{\alpha-1}+1 \ (\mathrm{mod}\ 2^\alpha) \text{ with } \alpha \geq 3$

$\phi_6 : \ s \equiv 1 + 2^{\alpha-1}, \ u \equiv 1 \ (\mathrm{mod}\ 2^\alpha) \quad (2^\alpha n^+, o(s, n^-)) = 1 \text{ or } 2$

$r \equiv -1$

$\phi_7 : \ u \equiv 1 \ (\mathrm{mod}\ 2^\alpha) \quad (n^+, o(s, 2^\alpha n^-)) = 1$

$r \equiv 2^{\alpha-1}\text{-}1 \ (\mathrm{mod}\ 2^\alpha) \text{ with } \alpha \geq 3$

$\phi_8 : \ u \equiv 1 \ (\mathrm{mod}\ 2^\alpha) \quad 4 \,|\, o(s, 2^\alpha n^-)$ and

$$1 + s + \cdots + s^{\frac{o(s,2^\alpha n^-)}{2} - 1} \equiv 2^{\alpha-1} n^- \ (\mathrm{mod}\ 2^\alpha n^-), \ (\mathrm{n}^+, o(s, 2^\alpha n^-)) = 1$$

Case 3. $\phi(a^j)=a^{js}, \ \phi(a^j b)=a^{js+u}b \quad (n,s)=1$

$\exists\, n_1^+, n_2^+$ s. t. $n^+ = n_1^+ n_2^+$ with $n_1^+, \ n_2^+ > 1, \ (n_1^+, \ n_2^+)=1$

$u \equiv 0 \ (\mathrm{mod}\ n_1^+), \ \ s \equiv u \equiv 1 \ (\mathrm{mod}\ n_2^+), \ \ u \equiv 1 \ (\mathrm{mod}\ n^-)$

$\exists\, m$ s. t. $s^m \equiv -1 \ (\mathrm{mod}\ n_1^+), \ \ (m, n_2^+) = 1, \ \ (2m, o(s, n^-)) = (m, o(s, n^-))$

$(n_2^+, o(s, n^-)) = 1$

$r \equiv 1$ or $2^{\alpha-1}+1 \ (\mathrm{mod}\ 2^\alpha)$

$\phi_9 : s \equiv u \equiv 1 \qquad \alpha = 0 \ \Rightarrow$ no more condition

$\alpha=1 \Rightarrow \ 4 | o(s, n_1^+ n^-) \qquad \alpha \geq 2 \ \Rightarrow \ o(s, n_1^+ n^-) \equiv 4 \ (\mathrm{mod}\ 8)$

$r \equiv 1$ or $2^{\alpha-1}+1 \ (\mathrm{mod}\ 2^\alpha)$ with $\alpha \geq 3$

$\phi_{10} : \ s \equiv 1 + 2^{\alpha-1}, \ u \equiv 1 \ (\mathrm{mod}\ 2^\alpha) \quad o(s, n_1^+ n^-) \equiv 4 \ (\mathrm{mod}\ 8)$

$r \equiv -1 \qquad \phi_{11} : \ u \equiv 1 \ (\mathrm{mod}\ 2^\alpha) \quad (n_2^+, o(s, 2^\alpha n_1^+ n^-)) = 1$

$r \equiv 2^{\alpha-1}$-1 $(\mathrm{mod}\ 2^\alpha)$ with $\alpha \geq 3$

$\phi_{12} : \ u \equiv 1 \ (\mathrm{mod}\ 2^\alpha) \quad \exists\, 2m_1$ s. t. $s^{2m_1} \equiv -1 \ (\mathrm{mod}\ n_1^+), \ (m_1, n_2^+) = 1$

and $1 + s + \cdots + s^{2m_1-1} \equiv 2^{\alpha-1} n^- \ (\mathrm{mod}\ 2^\alpha n^-)$

[AB-type]

In this case, $n^{+}=1$ and hence, $n=2^{\alpha}n^{-}$.

$$\phi(a^{2j})=a^{2js},\ \ \phi(a^{2j+1})=a^{2js+2u}b,\ \ \phi(a^{2j}\mathrm{b})=a^{2js+1},\ \ \phi(a^{2j+1}b)=a^{2js+2v+1}b$$

$\exists\, m$ s. t. $s^{2m}\equiv -1,\ s^{2}\equiv 2us+1,\ 2u\equiv 2v\ (\mathrm{mod}\ 2n^{-})$

Case 1. $r\equiv 1$ or $2^{\alpha-1}+1\ (\mathrm{mod}\ 2^{\alpha})$

$\phi_{13}:\ s\equiv 1,\ 2u\equiv 2,\ 2v\equiv 0\ (\mathrm{mod}\ 2^{\alpha})$

$\alpha\leq 1\Rightarrow$ any m, $\alpha\geq 2\Rightarrow m:\mathrm{odd}$

Case 2. $r\equiv 1$ or $2^{\alpha-1}+1\ (\mathrm{mod}\ 2^{\alpha})$ with $\alpha\geq 3$

$\phi_{14}:\ s\equiv 1,\ 2u\equiv 2+2^{\alpha-1},\ 2v\equiv 0\ (\mathrm{mod}\ 2^{\alpha})$ $m:\mathrm{odd}$

Case 3. $r\equiv -1$

$\phi_{15}:\ s\equiv 1,\ 2u\equiv 2v\equiv 2\ (\mathrm{mod}\ 2^{\alpha})$ with $\alpha=2,\ \mathrm{m}:\mathrm{odd}$

Case 4. $r\equiv 2^{\alpha-1}-1$

$\exists$ no AB-type t-bal. Cayley skew.

$[n : \text{odd or } 2n']$

$n = n^{+}$: $\phi_1,\ \phi_5,\ \phi_9$

$n = n^{-}$: $\phi_1,\ \phi_{13}$

$n = n^{+}n^{-}$ with $n^{+},\ n^{-} > 1$: $\phi_1,\ \phi_5,\ \phi_9$

$[n = 4n']$

$n' = n^{+},\ r \equiv 1 \pmod 4$: $\phi_2,\ \phi_5,\ \phi_9$

$n' = n^{+},\ r \equiv -1 \pmod 4$: $\phi_1,\ \phi_7,\ \phi_{11}$

$n' = n^{-},\ r \equiv 1 \pmod 4$: $\phi_2,\ \phi_{13}$

$n' = n^{-},\ r \equiv -1 \pmod 4$: $\phi_1,\ \phi_{15}$

$n' = n^{+}n^{-}$ with $n^{+},\ n^{-} > 1,\ r \equiv 1 \pmod 4$: $\phi_2,\ \phi_5,\ \phi_9$

$n' = n^{+}n^{-}$ with $n^{+},\ n^{-} > 1,\ r \equiv -1 \pmod 4$: $\phi_1,\ \phi_7,\ \phi_{11}$

$[n = 2^{\alpha} n' \text{ with } \alpha \geq 3]$

$n' = n^+,\ r \equiv 1 \text{ or } 2^{\alpha-1}+1$: $\phi_2,\ \phi_3,\ \phi_5,\ \phi_6,\ \phi_9,\ \phi_{10}$

$n' = n^+,\ r \equiv -1$: $\phi_1,\ \phi_7,\ \phi_{11}$

$n' = n^+,\ r \equiv 2^{\alpha-1}-1$: $\phi_4,\ \phi_8,\ \phi_{12}$

$n' = n^-,\ r \equiv 1 \text{ or } 2^{\alpha-1}+1$: $\phi_2,\ \phi_3,\ \phi_{13},\ \phi_{14}$

$n' = n^-,\ r \equiv -1$: $\phi_1,\ \phi_{15}$

$n' = n^-,\ r \equiv 2^{\alpha-1}-1$: ϕ_4

$n' = n^+ n^- \text{ with } n^+,\ n^- > 1,\ r \equiv 1 \text{ or } 2^{\alpha-1}+1$: $\phi_2,\ \phi_3,\ \phi_5,\ \phi_6,\ \phi_9,\ \phi_{10}$

$n' = n^+ n^- \text{ with } n^+,\ n^- > 1,\ r \equiv -1$: $\phi_1,\ \phi_7,\ \phi_{11}$

$n' = n^+ n^- \text{ with } n^+,\ n^- > 1,\ r \equiv 2^{\alpha-1}$-1: $\phi_4,\ \phi_8,\ \phi_{12}$

Future research

1. Classification of skew-morphisms of cyclic groups.

2. Classification of regular t-balanced Cayley maps on abelian groups.

3. Classification of smooth skew-morphisms of semidirect product of $\mathbb{Z}_n$ by $\mathbb{Z}_2$.

4. Complementary product of Γ and cyclic group

$\leftrightarrow$ skew-morphism of Γ

Complementary product of Γ and dihedral group

$\leftrightarrow$??? of Γ

Thank you!!!!