Competition Numbers and Phylogeny Numbers

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Outline

1. Definition
2. Results

Graphs

- $G = \big(V(G), E(G)\big)$, $E(G) \subseteq \binom{V(G)}{2}$;
- $G' \lhd G$: $V(G') \subseteq V(G)$ and $E(G') = E(G) \cap \binom{V(G')}{2}$; we call G' an induced subgraph of the graph G;
- $\mathcal{I}_k(G)$: the graph obtained from G by adding k isolated vertices.

Digraphs

- $D = \big(\mathrm{V}(D), \mathrm{A}(D)\big)$, $\mathrm{A}(D) \subseteq \mathrm{V}(D) \times \mathrm{V}(D)$;
- Out-neighborhood of v in D: $\mathrm{N}_D^+(v) = \{w : (v, w) \in \mathrm{A}(D)\}$;
- D°: $\mathrm{V}(D^\circ) = \mathrm{V}(D)$, $\mathrm{A}(D^\circ) = \mathrm{A}(D) \cup \{(v, v) : v \in \mathrm{V}(D)\}$.

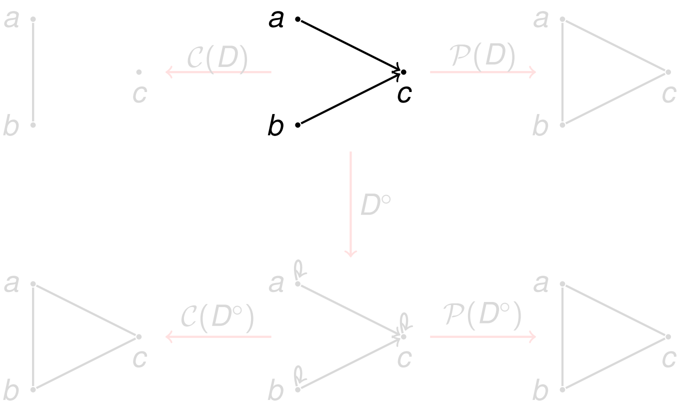
Competition/Phylogeny Graphs

Let D be a digraph.

The competition graph of D, denoted by $\mathcal{C}(D)$, is the graph with

- $\mathrm{V}(\mathcal{C}(D)) = \mathrm{V}(D)$
- $uv \in \mathrm{E}(\mathcal{C}(D))$ if $u \neq v$, $\mathrm{N}_D^+(u) \cap \mathrm{N}_D^+(v) \neq \emptyset$.

The phylogeny graph of D, denoted by $\mathcal{P}(D)$, is the competition graph of D°, namely $\mathcal{P}(D) = \mathcal{C}(D^\circ)$.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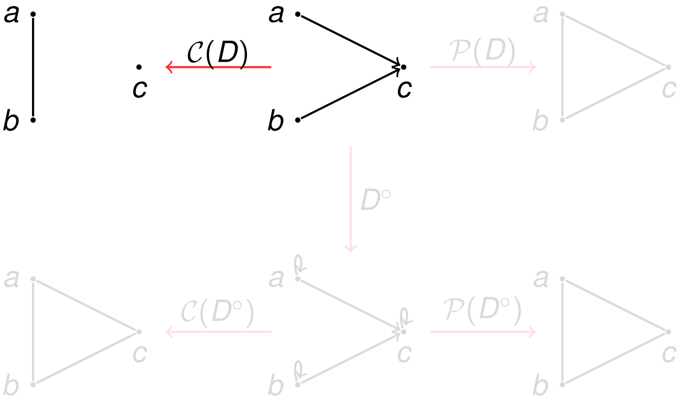
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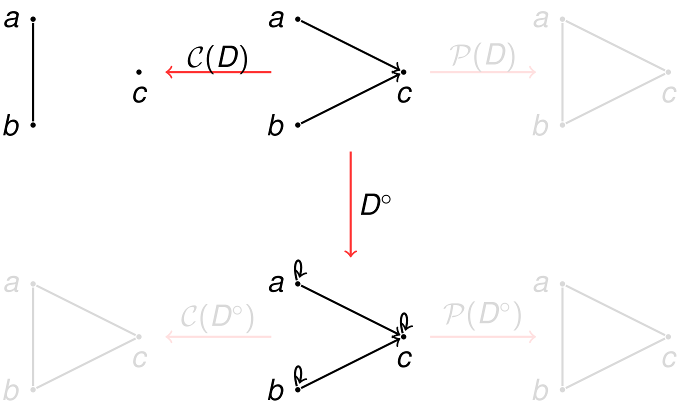
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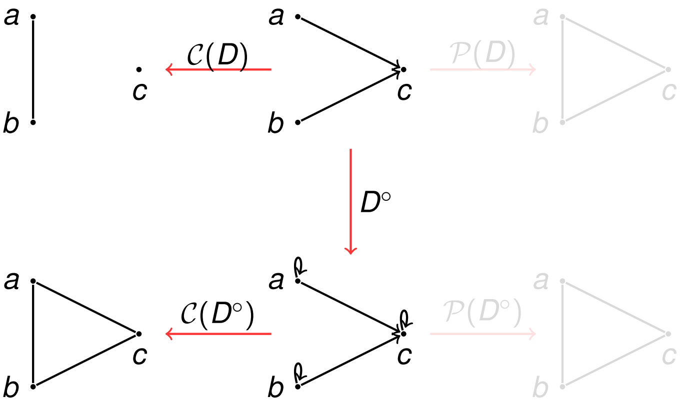
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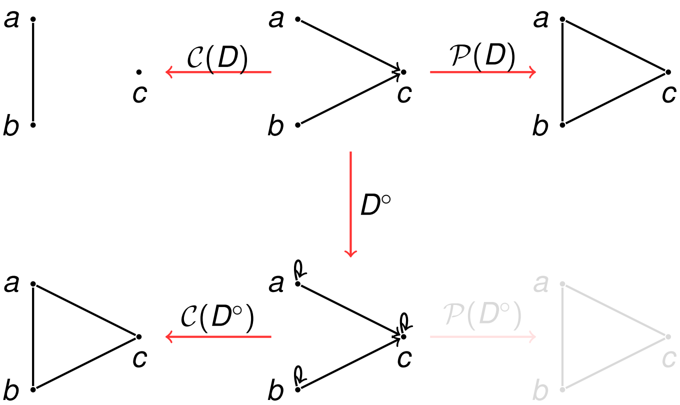
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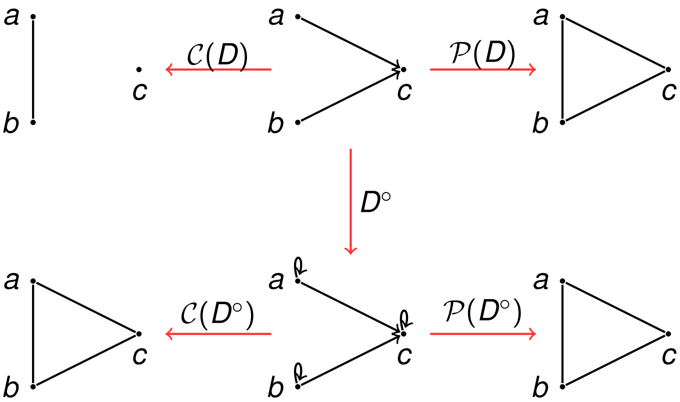
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Competition/Phylogeny Numbers

Given a graph G,
the competition number of G,

- $\kappa(G) = \min\{|\mathrm{V}(D)| - |\mathrm{V}(G)| :$
  $G \lhd \mathcal{C}(D)$ for some acyclic digraph $D\}$

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- $\phi(G) = \min\{|\mathrm{V}(D)| - |\mathrm{V}(G)| :$
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Equivalent definition:

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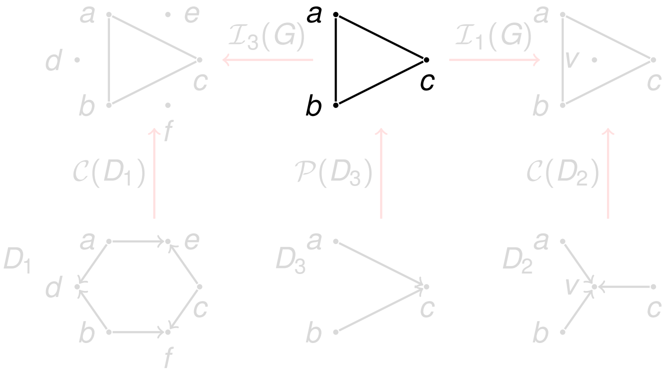

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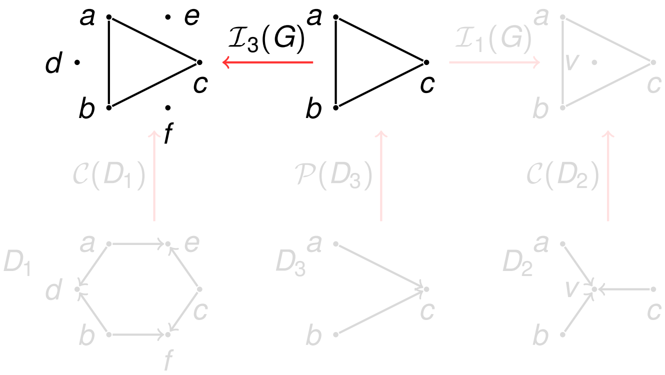

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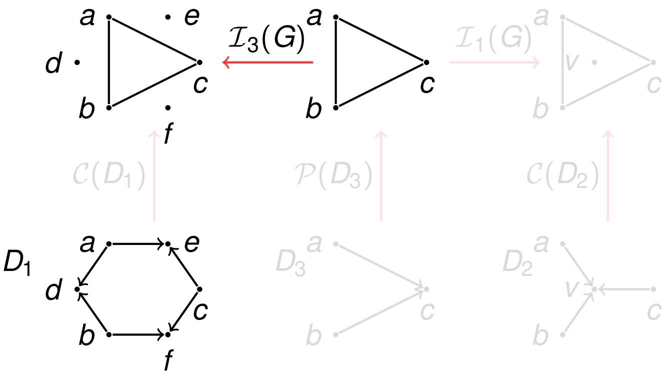

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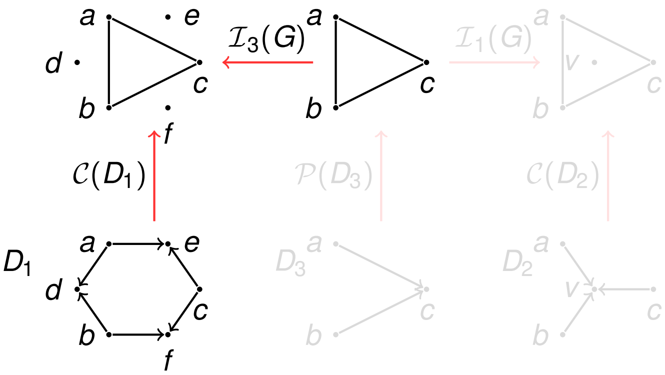

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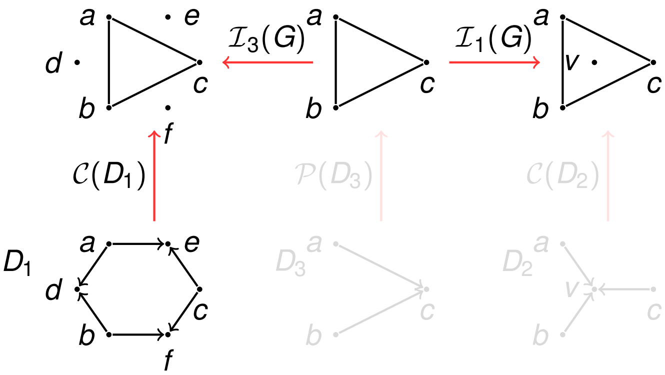

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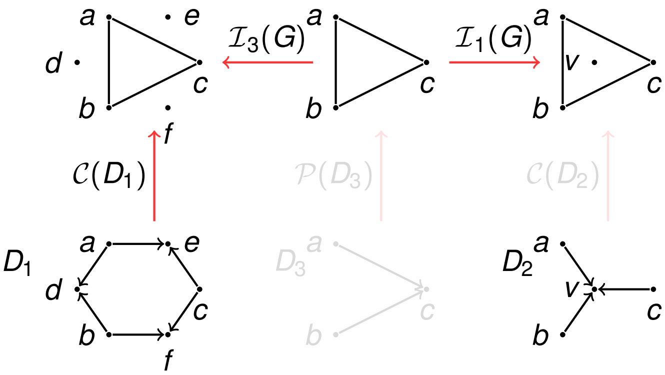

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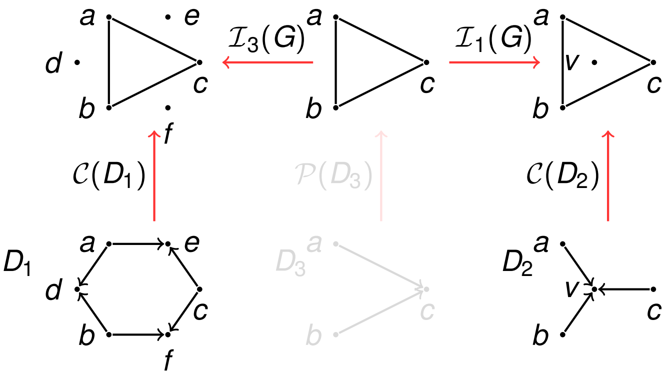

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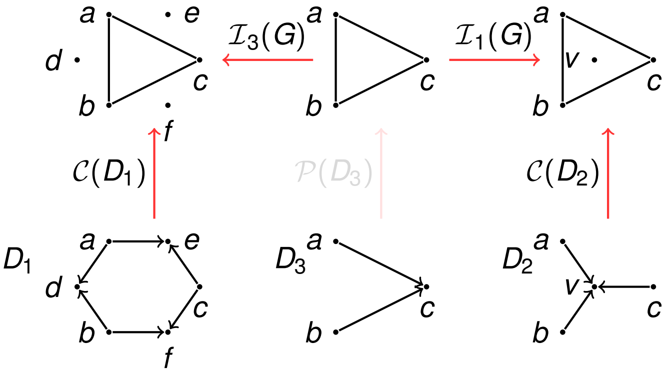

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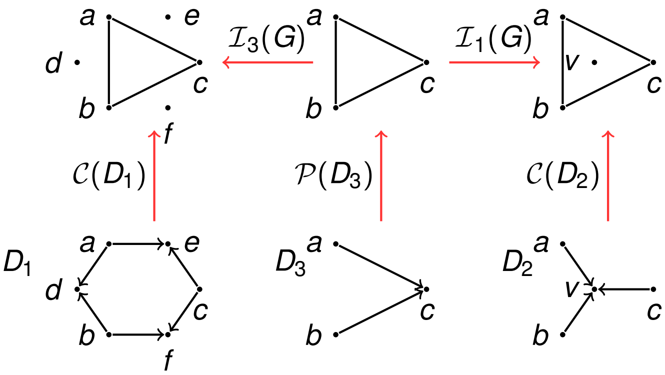

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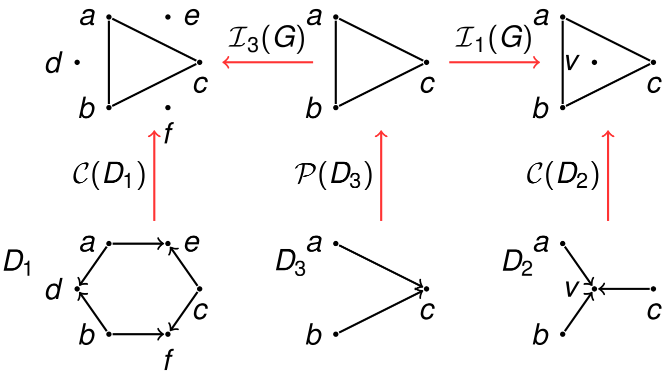

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The competition/phylogeny number is a measure of the distance between a graph and the class of competition/phylogeny graphs of acyclic digraphs.

The competition/phylogeny number of a finite graph must be finite.

In 1986, R. J. Opsut proved that it is NP-complete to calculate $\kappa(G)$.

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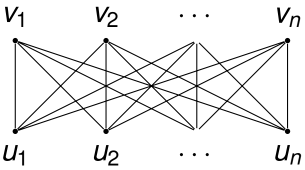
Uniform Multipartite Graphs

Denote K_m^n to be the uniform complete multipartite graph with m parts and uniform part size n such that two vertices are adjacent if and only if they are in different parts.



K_2^n

Theorem

1. *(2008 Kim-Park-Sano) For* $m \geq 2$, $\kappa(K_m^2) = 2$;
2. *(2008 Kim-Park-Sano) For* $m \geq 3$, $\kappa(K_m^3) = 4$;
3. *(2008 Kim-Sano) For* $n \geq 2$, $\kappa(K_3^n) = n^2 - 3n + 4$;
4. *(2009 Kim-Park-Sano) For* $n \geq 4$,

$$n^2 - 4n + 6 \leq \kappa(K_m^4) \leq n^2 - 4n + 8;$$

5. *(2012 Li-Chang) For* $2 \leq m \leq \mathcal{L}(n) + 2$,

$$\kappa(K_m^n) \leq n^2 - 2n + 2;$$

6. *(2012 Kim-Park-Sano) For* $3 \leq n \leq \mathcal{L}(n) + 2, m \geq n$,

$$\kappa(K_m^n) \leq n^2 - n + 1.$$

Theorem (2018+ Wu-X-Zaw)

If $\mathcal{L}(n) = n - 1, m \geq 1$, *then*

$$\kappa(K_m^n) \leq n^2 - 2n + 2.$$

Lemma (2018+ Wu-X-Zaw)

For every graph G, it holds

$$\phi(G) - \kappa(G) + 1 \geq 0.$$

What are those graphs G with $\phi(G) - \kappa(G) + 1 = 0$?

Figure: $\phi(G) - \kappa(G) + 1 = 0 - 1 + 1 = 0$

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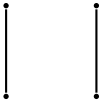

Figure: $\phi(G) - \kappa(G) + 1 = 0 - 1 + 1 = 0$

Theorem (2018+ Wu-X-Zaw)

For the following graphs G, it holds $\phi(G) - \kappa(G) + 1 = 0$.

1. K_m^2, *for* $m \geq 2$;
2. K_m^3, *for* $m \geq 3$;
3. K_3^n, *for* $n \geq 2$;
4. *Connected graphs with at most one triangle.*

Question: Is it true that

$$\phi(G) - \kappa(G) + 1 = 0$$

for all connected graphs G?

NO!

Theorem (2018+ X-Zaw-Zhu)

For every integer k, there exists a connected graph G satisfying

$$\phi(G) - \kappa(G) + 1 > k.$$

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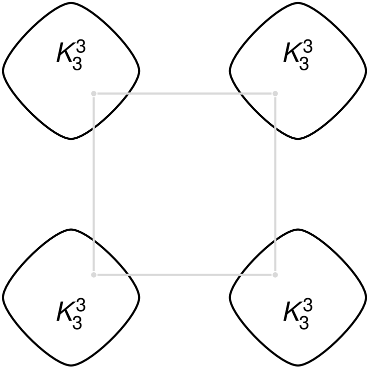
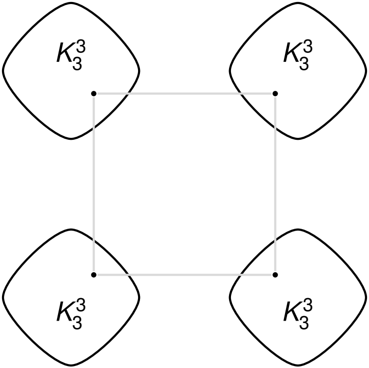


Figure: G_4

$$\phi(G_m) - \kappa(G_m) + 1 > m, \text{ for } m \geq 4.$$



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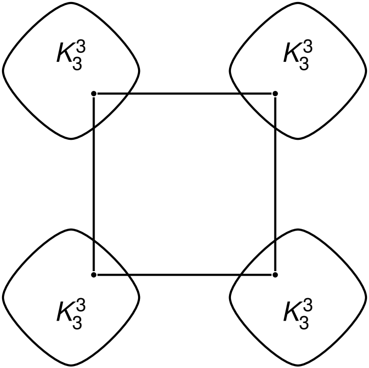

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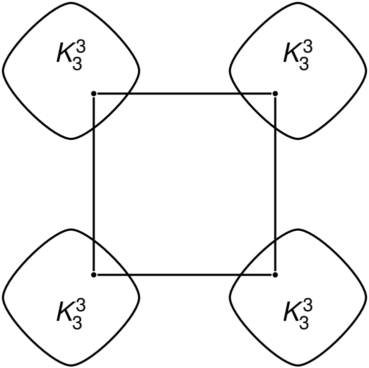

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Theorem (2018+ Wu-X-Zaw)

For every integer k, there exists a graph G such that $\phi(G) - \kappa(G) + 1 = k$ if and only if $k \geq 0$.

Problem (open)

Is it true that for every given positive integer k, there exists a connected graph G such that

$$\phi(G) - \kappa(G) + 1 = k?$$

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Thank you for your attention!