Relative t-designs on one shell of Johnson association schemes

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Outline

Association scheme

Let X be a finite set with $|X| = n$ and $\mathcal{R} = \{R_0, R_1, \ldots, R_d\} \subseteq X \times X$.
The adjacency matrix of (X, R_i) is given by

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } (x, y) \in R_i \\ 0, & \text{otherwise.} \end{cases}$$

$\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ is called a symmetric association scheme of class d if

- $A_0 = I$.
- $\sum_{i=0}^{d} A_i = J$.
- ${}^t A_i = A_i$ for $1 \le i \le d$.
- $A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k$.

Q-polynomial association scheme

$\mathcal{A} = \mathrm{span}\{A_0, A_1, \ldots, A_d\}$ is called Bose-Mesner algebra of $\mathfrak{X}$.
Since $\mathfrak{X}$ is symmetric, $\mathcal{A}$ has another basis called primitive idempotents $\{E_0, E_1, \ldots, E_d\}$ with respect to entry-wise multiplication.

- $E_0 = \frac{1}{|X|} J$.
- $\sum_{i=0}^{d} E_i = I$.
- $E_i E_j = \delta_{i,j} E_i$.
- $E_i \circ E_j = \sum_{k=0}^{d} q_{i,j}^{k} E_k$.

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A symmetric association scheme $\mathfrak{X}$ is called Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_d$, if there exists a polynomial $u_j(x)$ of degree j such that $E_j = u_j(E_1^{\circ})$, where E_1° means entry-wise multiplication.

Johnson association scheme

Given positive integers v, k with $v \geq 2k$, let $V = \{1, 2, \cdots, v\}$ and $X = \binom{V}{k}$. Define

$$R_r = \{(x, y) \in X \times X : |x \cap y| = k - r\}.$$

Then $J(v, k) = (X, \{R_r\}_{r=0}^{k})$ is Johnson association scheme.

- It is known that $J(v, k)$ is a Q-polynomial association scheme.

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For a fixed point $u_0 = \{1, 2, \cdots, k\}$, the r-th shell of $J(v, k)$ is defined by

$$X_r := \{x : |x \cap u_0| = k - r\}.$$

Relative t-designs in Q-polynomial A.S.

Let $u_0 \in X$ be a fixed point and (Y, w) be a weighted subset of X. Define

$$\chi_{(Y,w)}(y) = \begin{cases} w(y) & \text{if } y \in Y, \\ 0 & \text{if } y \notin Y. \end{cases}$$

Definition (Delsarte, 1977)

Let $\mathfrak{X}$ be a Q-polynomial association scheme. A weighted subset (Y, w) of X is called a relative t-design in $\mathfrak{X}$ with respect to u_0 if $E_j\chi_{(Y,w)}$ and $E_j\chi_{\{u_0\}}$ are linearly dependent for all $1 \leq j \leq t$.

Definition (Delsarte, 1973)

Let $\mathfrak{X}$ be a Q-polynomial association scheme. A weighted subset (Y, w) is called a t-design in $\mathfrak{X}$ $E_j\chi_{(Y,w)} = 0$ for all $1 \leq j \leq t$.

Block designs and the generalization

A t-(v,k,λ) design $(V,\mathcal{B})$, or t-design in $J(v,k)$ consists of sets of

- points: V with $|V| = v$,
- blocks: non-empty subset $\mathcal{B}$ of $X = \binom{V}{k}$,

so that for every $T \in \binom{V}{t}$,

$$\#\{B \in \mathcal{B} : T \subseteq B\} = \lambda > 0.$$

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$$\#\{B \in \mathcal{B} : T \subseteq B\} = \lambda > 0.$$

- $w : \mathcal{B} \to \mathbb{R}_{>0}$.
- $X = \binom{V}{k_1} \cup \binom{V}{k_2} \cdots \cup \binom{V}{k_p}$.

$(V, \mathcal{B}, w)$ is called a weighted regular t-wise balanced design if

$$\sum_{B \in \mathcal{B}, T \subseteq B} w(B) = \lambda > 0.$$

$\iff$ A relative t-design in $H(v, 2)$ with with respect to $(0, 0, \cdots, 0)$.

Our problem

t-design in $J(v,k)$

$\downarrow$ *Delsarte* $\searrow$

relative t-designs in $H(v,2)$

$Y \subset \binom{V}{k_1} \cup \cdots \cup \binom{V}{k_p}$

$\Downarrow$

$Y \cap \binom{V}{k_i}$: $(t+1-p)$-design in $J(v,k_i)$

relative t-designs in $J(v,k)$

$X_{r_1} \cup \cdots \cup X_{r_p}$

$\Downarrow$

Question

Our problem

t-design in $J(v,k)$

$\downarrow$ $\quad$ $\xrightarrow{\textit{Delsarte}}$

relative t-designs in $H(v,2)$: $Y \subset \binom{V}{k_1} \cup \cdots \cup \binom{V}{k_p}$

relative t-designs in $J(v,k)$: $X_{r_1} \cup \cdots \cup X_{r_p}$

$\Downarrow$

$Y \cap \binom{V}{k_i}$: $(t+1-p)$-design in $J(v,k_i)$

$\Downarrow$

Question

Question: Is $Y \cap X_{r_i}$ 'some' design in X_{r_i}?

- Structure of one shell X_r of $J(v,k)$.
- How to define designs in X_r?

Product association scheme

Let $(Y_\ell, \mathcal{A}_\ell)$ be an association scheme of class k_ℓ with Bose-Mesner Algebra $\mathcal{A}_\ell$. The direct product of m number of association schemes is

$$(X, \mathcal{A}) = (Y_1, \mathcal{A}_1) \otimes (Y_2, \mathcal{A}_2) \otimes \cdots \otimes (Y_m, \mathcal{A}_m),$$

where

$$X = Y_1 \times Y_2 \times \cdots \times Y_m$$

$$\mathcal{A} = \{\otimes_{\ell=1}^{m} B_\ell \mid B_\ell \in \mathcal{A}_\ell, 1 \leq \ell \leq m\}.$$

Structure of one shell of $J(v, k)$

Recall that $u_0 = \{1, 2, \ldots, k\}$ and $X_r = \left\{ x \in \binom{V}{k} : |x \cap u_0| = k - r \right\}$.

1. If $2 \leq r \leq \frac{k}{2}$, take $X_r := \{(u_0 - x, x - u_0) \mid x \in X_r\}$, i.e.,

$$X_r = J(k, r) \otimes J(v - k, r).$$

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2. If $\frac{k}{2} < r \leq \frac{v-k}{2}$, take $X_r := \{(x \cap u_0, x - u_0) \mid x \in X_r\}$, i.e.,
$$X_r = J(k, k - r) \otimes J(v - k, r).$$
3. If $\frac{v-k}{2} < r \leq k - 2$, take $X_r := \{(x \cap u_0, (V - u_0) - x \mid x \in X_r\}$, i.e.,
$$X_r = J(k, k - r) \otimes J(v - k, v - k - r).$$

Structure of one shell of $J(v,k)$

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3. If $\frac{v-k}{2} < r \le k-2$, take $X_r := \{(x \cap u_0, (V - u_0) - x \mid x \in X_r\}$, i.e.,

$$X_r = J(k,k-r) \otimes J(v-k,v-k-r).$$

Remark: The product association scheme $J(v_1,k_1) \otimes J(v_2,k_2)$ is a commutative, but not Q-polynomial association scheme.

Designs in product association schemes

Designs in in general product association schemes are defined using the primitive idempotent for each component.

Definition (Martin, 1998)

Let $|V_i| = v_i$ for $i = 1, 2$ and $X = \binom{V_1}{k_1} \times \binom{V_2}{k_2}$. A weighted subset (Y, w) of X is called a weighted t-design in $J(v_1, k_1) \otimes J(v_2, k_2)$ if for $j_1 + j_2 = t$

$$\sum_{\substack{(y_1,y_2)\in Y \\ z_1\subseteq y_1, z_2\subseteq y_2}} w(y_1, y_2) = \lambda_{(j_1,j_2)}$$

is independent on the choice of $(z_1, z_2) \in \binom{V_1}{j_1} \times \binom{V_2}{j_2}$.

In particular, it is called a mixed t-design if $w = 1$.

Main result

- (Y, w) is supported by p shells if $\{r_1, \ldots, r_p\} = \{r \mid Y \cap X_r \neq \emptyset\}$.
- Recall that the r-th shell of $J(v, k)$ is $X_r = J(k, k_1) \otimes J(v - k, k_2)$

Theorem (Bannai-Z., 2018)

If (Y, w) is a relative t-design in $J(v, k)$ on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$, then $(Y \cap X_{r_i}, w)$ is a weighted $(t + 1 - p)$-design in X_{r_i} as a product association scheme for $1 \leq i \leq p$.

Lower bound for relative t-designs on one shell

Theorem (Bannai-Bannai-Suda-Tanaka, 2015; Martin, 1998)

Let Y be a t-design in one shell X_r of $J(v,k)$. Then

$$|Y| \geq \begin{cases} \binom{v}{e} - \binom{v}{e-1} & \text{if} \quad t = 2e, \\ 2\left(\binom{v-1}{e} - \binom{v-1}{e-1}\right) & \text{if} \quad t = 2e+1. \end{cases}$$

The design (Y, w) is called tight if the above lower bound is attained.

Tight 2-designs in X_r

All known tight mixed 2-designs in X_r come from symmetric 2-designs with one block removed. More precisely, $(V, \mathcal{B})$ is a symmetric 2-(v, k, λ) design. Let V_1 be the points set of a block $B \in \mathcal{B}$ and $V_2 = V \backslash V_1$, then $(V_1 \times V_2, \mathcal{B} \backslash B)$ forms a mixed 2-design with $\lambda_{(2,0)} + 1 = \lambda_{(1,1)} = \lambda_{(0,2)} = \lambda$.

points

	1	1	1	0	0	0	0
blocks	1	0	0	1	1	0	0
	1	0	0	0	0	1	1
	0	1	0	1	0	0	1
	0	1	0	1	1	0	0
	0	0	1	1	0	1	0
	0	0	1	0	1	0	1
		V_1			V_2		

2-(7,3,1) design gives a 2-design in $J(3,1) \otimes J(4,2)$.

Tight 3-designs in X_r

Possible parameters of tight 3-designs in X_r for $v \leq 1,000$ are of type:

$$v = 4u,\ k = 2u,\ k_1 = k_2 = r, |Y| = 4(2u-1), \quad \text{for } 2 \leq r \leq u.$$

Construction:

1. H: symmetric Hadamard 2-$(4u-1, 2u-1, u-1)$ design

$H=$

${}^t\mathbf{1}_{2u-1}$	${}^t\mathbf{0}_{2u}$
H_{Ind}	H_{Res}

H_{Ind}: 2-$(2u-1, u-1, u-2)$ design

H_{Res}: 2-$(2u, u, u-1)$ design.

2. Tight 3-designs in $J(2u, u) \otimes J(2u, u)$

$\mathbf{1}_{4u-2}$	H_{Ind}	H_{Res}
$\mathbf{0}_{4u-2}$	H^c_{Ind}	H^c_{Res}

Further work

- If (Y, w) is a relative t-design in a Q-polynomial association scheme on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$ with respect to a fixed point, then for each i whether $Y \cap X_{r_i}$ is some design in X_{r_i} ?

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- The same question for P-polynomial association scheme.

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- The same question for P-polynomial association scheme.

Both of the problems are solved for $H(v, 2)$ and $J(v, k)$.

Thank you for your attention.