Relative *t*-designs on one shell of Johnson association schemes

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Outline

- **I** Relative *t*-designs in Q-polynomial association scheme
- 2 Designs in product association scheme
- 3 Main theorem
- 4 Tight relative t-designs on one shell

Association scheme

Let X be a finite set with |X| = n and $\mathcal{R} = \{R_0, R_1, \dots, R_d\} \subseteq X \times X$. The adjacency matrix of (X, R_i) is given by

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } (x, y) \in R_i \\ 0, & \text{otherwise.} \end{cases}$$

 $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ is called a symmetric association scheme of class d if $A_0 = I$.

$$\bullet \sum_{i=0}^{d} A_i = J.$$

•
$${}^{t}A_{i} = A_{i}$$
 for $1 \leq i \leq d$.

$$\bullet A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k.$$

Q-polynomial association scheme

 $\mathcal{A} = \text{span}\{A_0, A_1, \dots, A_d\}$ is called Bose-Mesner algebra of \mathfrak{X} . Since \mathfrak{X} is symmetric, \mathcal{A} has another basis called primitive idempotents $\{E_0, E_1, \dots, E_d\}$ with respect to entry-wise multiplication.

- $E_0 = \frac{1}{|X|}J.$
- $\bullet \sum_{i=0}^{d} E_i = I.$
- $\bullet E_i E_j = \delta_{i,j} E_i.$
- $E_i \circ E_j = \sum_{k=0}^d q_{i,j}^k E_k.$

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A symmetric association scheme \mathfrak{X} is called Q-polynomial with respect to the ordering E_0, E_1, \ldots, E_d , if there exists a polynomial $u_j(x)$ of degree j such that $E_j = u_j(E_1^\circ)$, where E_1° means entry-wise multiplication.

Johnson association scheme

Given positive integers v, k with $v \ge 2k$, let $V = \{1, 2, \dots, v\}$ and $X = \binom{V}{k}$. Define

$$R_r = \{(x, y) \in X \times X : |x \cap y| = k - r\}.$$

Then $J(v, k) = (X, \{R_r\}_{r=0}^k)$ is Johnson association scheme.

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For a fixed point $u_0 = \{1, 2, \dots, k\}$, the *r*-th shell of J(v, k) is defined by

 $X_r := \{x : |x \cap u_0| = k - r\}.$

Relative *t*-designs in Q-polynomial A.S.

Let $u_0 \in X$ be a fixed point and (Y, w) be a weighted subset of X. Define

$$\chi_{(Y,w)}(y) = \begin{cases} w(y) & \text{if } y \in Y, \\ 0 & \text{if } y \notin Y. \end{cases}$$

Definition (Delsarte, 1977)

Let \mathfrak{X} be a Q-polynomial association scheme. A weighted subset (Y, w) of X is called a relative *t*-design in \mathfrak{X} with respect to u_0 if $E_{j\chi(Y,w)}$ and $E_{j\chi_{\{u_0\}}}$ are linearly dependent for all $1 \leq j \leq t$.

Definition (Delsarte, 1973)

Let \mathfrak{X} be a Q-polynomial association scheme. A weighted subset (Y, w) is called a *t*-design in $\mathfrak{X} \in E_{jX(Y,w)} = 0$ for all $1 \le j \le t$.

Block designs and the generalization

A t-(v, k, λ) design (V, B), or t-design in J(v, k) consists of sets of

• points: V with |V| = v,

• blocks: non-empty subset \mathcal{B} of $X = \binom{V}{k}$,

so that for every $T \in \binom{V}{t}$,

 $\#\{B\in \mathcal{B}: T\subseteq B\}=\lambda>0.$

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•
$$w: \mathcal{B} \to \mathbb{R}_{>0}$$
.

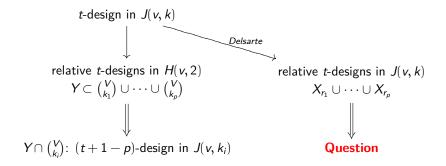
•
$$X = \binom{V}{k_1} \cup \binom{V}{k_2} \cdots \cup \binom{V}{k_p}$$
.

 (V, \mathcal{B}, w) is called a weighted regular *t*-wise balanced design if

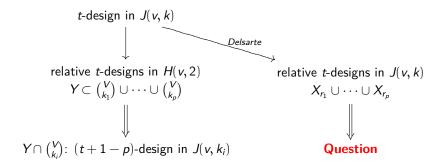
$$\sum_{B\in\mathcal{B}, T\subseteq B} w(B) = \lambda > 0.$$

 \iff A relative *t*-design in H(v, 2) with with respect to $(0, 0, \cdots, 0)$.

Our problem



Our problem



Question: Is $Y \cap X_{r_i}$ 'some' design in X_{r_i} ?

- Structure of one shell X_r of J(v, k).
- How to define designs in X_r ?

Product association scheme

Let (Y_{ℓ}, A_{ℓ}) be an association scheme of class k_{ℓ} with Bose-Mesner Algebra A_{ℓ} . The direct product of *m* number of association schemes is

 $(X, \mathcal{A}) = (Y_1, \mathcal{A}_1) \otimes (Y_2, \mathcal{A}_2) \otimes \cdots \otimes (Y_m, \mathcal{A}_m),$

where

$$X = Y_1 imes Y_2 imes \cdots imes Y_m$$

 $\mathcal{A} = \{ \otimes_{\ell=1}^m B_\ell \mid B_\ell \in \mathcal{A}_\ell, 1 \le \ell \le m \}.$

Structure of one shell of J(v, k)

Recall that
$$u_0=\{1,2,\ldots,k\}$$
 and $X_r=\Big\{x\in \binom{V}{k}:|x\cap u_0|=k-r\Big\}.$

1 If
$$2 \le r \le \frac{k}{2}$$
, take $X_r := \{(u_0 - x, x - u_0) \mid x \in X_r\}$, i.e.,

$$X_r = J(k,r) \otimes J(v-k,r).$$

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2 If $\frac{k}{2} < r \le \frac{v-k}{2}$, take $X_r := \{(x \cap u_0, x - u_0) \mid x \in X_r\}$, i.e., $X_r = J(k, k - r) \otimes J(v - k, r)$. 3 If $\frac{v-k}{2} < r \le k - 2$, take $X_r := \{(x \cap u_0, (V - u_0) - x \mid x \in X_r\}$, i.e.,

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Remark: The product association scheme $J(v_1, k_1) \otimes J(v_2, k_2)$ is a commutative, but not Q-polynomial association scheme.

Designs in product association schemes

Designs in in general product association schemes are defined using the primitive idempotent for each component.

Definition (Martin, 1998)

Let $|V_i| = v_i$ for i = 1, 2 and $X = \binom{V_1}{k_1} \times \binom{V_2}{k_2}$. A weighted subset (Y, w) of X is called a weighted t-design in $J(v_1, k_1) \otimes J(v_2, k_2)$ if for $j_1 + j_2 = t$

 $\sum_{\substack{(y_1, y_2) \in Y \\ z_1 \subseteq y_1, z_2 \subseteq y_2}} w(y_1, y_2) = \lambda_{(j_1, j_2)}$

is independent on the choice of $(z_1, z_2) \in {\binom{V_1}{j_1}} \times {\binom{V_2}{j_2}}$.

In particular, it is called a mixed *t*-design if w = 1.

Main result

- (Y, w) is supported by p shells if $\{r_1, \ldots, r_p\} = \{r \mid Y \cap X_r \neq \emptyset\}$.
- Recall that the *r*-th shell of J(v, k) is $X_r = J(k, k_1) \otimes J(v k, k_2)$

Theorem (Bannai-Z., 2018)

If (Y, w) is a relative t-design in J(v, k) on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$, then $(Y \cap X_{r_i}, w)$ is a weighted (t + 1 - p)-design in X_{r_i} as a product association scheme for $1 \le i \le p$.

Lower bound for relative *t*-designs on one shell

Theorem (Bannai-Bannai-Suda-Tanaka, 2015; Martin, 1998)

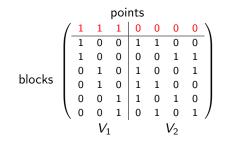
Let Y be a t-design in one shell X_r of J(v, k). Then

$$|\mathsf{Y}| \geq \begin{cases} \binom{\mathsf{v}}{e} - \binom{\mathsf{v}}{e-1} & \text{if } t = 2e, \\ 2\left(\binom{\mathsf{v}-1}{e} - \binom{\mathsf{v}-1}{e-1}\right) & \text{if } t = 2e+1. \end{cases}$$

The design (Y, w) is called tight if the above lower bound is attained.

Tight 2-designs in X_r

All known tight mixed 2-designs in X_r come from symmetric 2-designs with one block removed. More precisely, (V, B) is a symmetric 2- (v, k, λ) design. Let V_1 be the points set of a block $B \in B$ and $V_2 = V \setminus V_1$, then $(V_1 \times V_2, B \setminus B)$ forms a mixed 2-design with $\lambda_{(2,0)} + 1 = \lambda_{(1,1)} = \lambda_{(0,2)} = \lambda$.



2-(7,3,1) design gives a 2-design in $J(3,1) \otimes J(4,2)$.

Tight 3-designs in X_r

Possible parameters of tight 3-designs in X_r for $v \le 1,000$ are of type:

$$v = 4u, \ k = 2u, \ k_1 = k_2 = r, |Y| = 4(2u - 1), \ \text{ for } 2 \le r \le u.$$

Construction:

1 *H*: symmetric Hadamard 2 - (4u - 1, 2u - 1, u - 1) design



 H_{Ind} : 2-(2u-1, u-1, u-2) design

2 Tight 3-designs in $J(2u, u) \otimes J(2u, u)$

1_{4u-2}	H _{Ind}	H _{Res}
0 _{4<i>u</i>-2}	H ^c Ind	H^c_{Res}

Further work

• If (Y, w) is a relative *t*-design in a Q-polynomial association scheme on p shells $X_{r_1} \cup \cdots \cup X_{r_p}$ with respect to a fixed point, then for each i whether $Y \cap X_{r_i}$ is some design in X_{r_i} ?

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- The same question for P-polynomial association scheme.

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- The same question for P-polynomial association scheme.

Both of the problems are solved for H(v, 2) and J(v, k).

Thank you for your attention.