Neumaier Graphs

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G2R2, August 2018

Contents

Outline

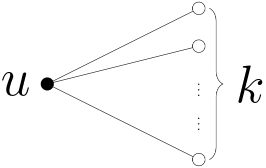
ERGs

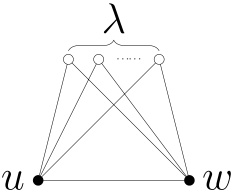
Definition

A graph Γ is *edge-regular* with *parameters* (v, k, λ) if it has v vertices, is non-empty, and

(i) each vertex is adjacent to exactly k vertices;

(ii) each pair of adjacent vertices have exactly λ common neighbours.



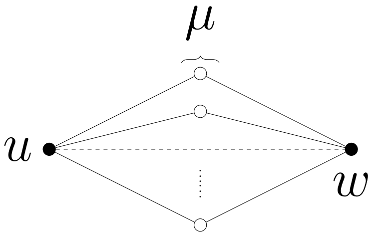


SRGs

Definition

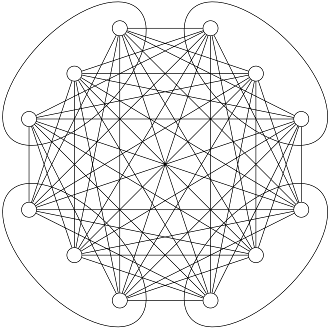
A graph Γ is *strongly regular* with *parameters* (v, k, λ, μ) if it is edge-regular with parameters (v, k, λ), non-complete, and

(iii) each pair of distinct non-adjacent vertices have exactly μ common neighbours.



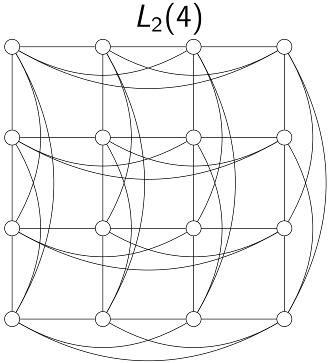
Examples of SRGs

$K_{4\times 3}$



Complete multipartite graph $K_{s\times t}$ is strongly regular, with parameters $(st, (s-1)t, (s-2)t, (s-1)t)$.

Examples of SRGs



Square lattice graph $L_2(s)$ is strongly regular, with parameters $(s^2, 2(s-1), s-2, 2)$

Regular Cliques

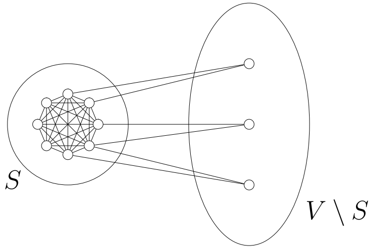
Definition

A *clique* in a graph Γ is a set of pairwise adjacent vertices. A clique of size s is called an *s-clique*.

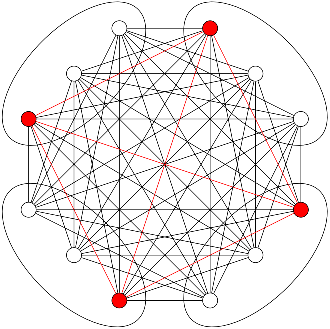
Regular Cliques

Definition

A clique S of Γ is *regular* if every vertex of Γ not in S is adjacent to the same number $m > 0$ of vertices in S. In this case we say that S has *nexus* m and is *m-regular*.

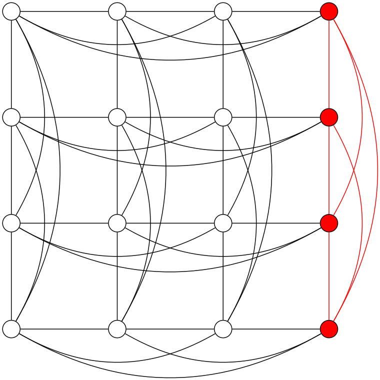


Examples of regular cliques



$(s-1)$-regular s-clique in $K_{s\times t}$.

Examples of regular cliques



1-regular s-clique in $L_2(s)$.

Outline

Neumaier's work

In the early 1980's, Arnold Neumaier studied regular cliques in ERGs, and certain finite geometries giving rise to certain SRGs with regular cliques. In particular, he proved the following

Theorem (Neumaier,1981)

A non-empty graph which is vertex-transitive, edge-transitive and contains a regular clique is complete or strongly regular.

Neumaier's question

Question

Is every edge-regular graph with a regular clique strongly regular?

Definition

- A *Neumaier graph* is a graph which is edge-regular, non-complete and contains a regular clique. We denote by $NG(v, k, \lambda; m, s)$ the set of Neumaier graphs which are edge-regular with parameters (v, k, λ) and contain an m-regular s-clique.
- A *strictly Neumaier graph* is a non-strongly regular Neumaier graph.

Answers and recent constructions

Goryainov and Shalaginov (2014) find 4 strictly Neumaier graphs in $NG(24, 8, 2; 1, 4)$.

Theorem (Greaves and Koolen, 2017)

There exist infinitely many strictly Neumaier graphs containing a 1-regular clique.

All of these are Cayley graphs, and their vertices can be partitioned into regular cliques.

Questions posed by Greaves and Koolen

Question 1

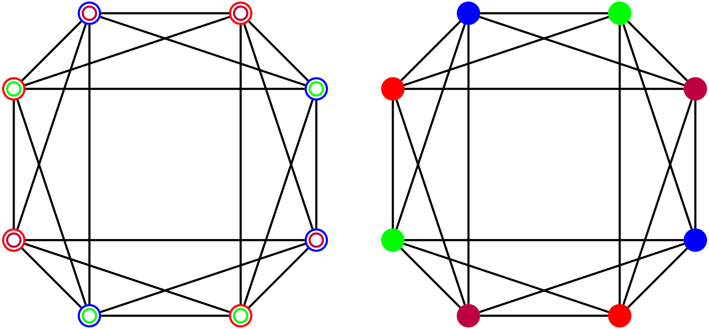
What is the minimum number of vertices for which there exists a strictly Neumaier graph?

Question 2

Does there exist a strictly Neumaier graph having a regular clique with nexus greater than 1?

Outline

Vertex-transitive Neumaier graphs



A graph in $NG(16, 9, 4; 2, 4)$, independently discovered by RJE, and Goryainov and Panasenko.
RJE used the classification of vertex-transitive graphs of order ≤ 47 by Holt and Royle.

Possible parameters for small Neumaier graphs

Question (2) is answered, so what about Question (1)?

Theorem (EGP)

(i) *Each strictly Neumaier graph with at most* 16 *vertices must be in* $NG(16, 9, 4; 2, 4)$.

(ii) *There is a unique strictly Neumaier graph (up to isomorphism) in* $NG(16, 9, 4; 2, 4)$.

To answer (i), we consider all possible tuples of non-negative integers v, k, λ with $v \leq 16$, such that an ERG with parameters (v, k, λ) could exist and contain a regular clique.

Valid edge-regular graph parameters

Lemma

Suppose Γ *is a non-complete edge-regular graph with parameters* (v, k, λ)*. Then:*

(i) $0 < k < v - 1, \; 0 \leq \lambda < k$.

(ii) $2 | vk, \; 2 | k\lambda, \; 6 | vk\lambda$.

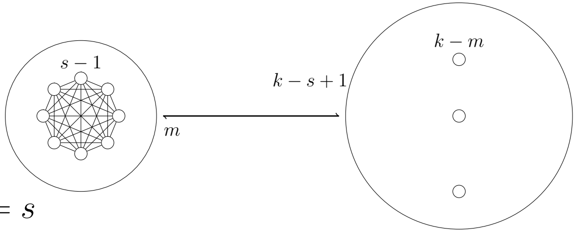
Lemma

Suppose Γ *is a non-complete edge-regular graph with parameters* (v, k, λ) *that contains an* m*-regular* s*-clique. Then:*

(i) $(v - s)m = (k - s + 1)s$.

(ii) $(k - s + 1)(s - 1) = (\lambda - s + 2)(m - 1)$.

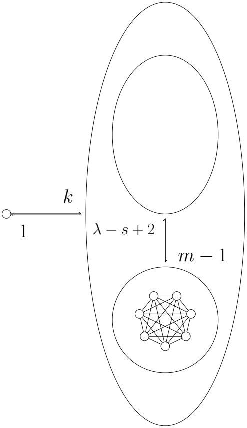
Counting arguments



$|S| = s$

$|V \setminus S| = v - s$

Counting arguments



A table of possible parameters

v	k	λ	s	m
4	2	0	2	1
6	3	0	2	1
	4	2	3	2
8	4	0	2	1
	6	4	4	3
9	4	1	3	1
	6	3	3	2
10	5	0	2	1
	6	3	4	2
	8	6	5	4
12	5	2	4	1
	6	0	2	1
		4	6	1
	8	4	3	2
	9	6	4	3
	10	8	6	5

v	k	λ	s	m
14	7	0	2	1
	9	6	7	3
	12	10	7	6
15	6	1	3	1
		3	5	1
	8	4	5	2
	10	5	3	2
		6	5	3
	12	9	5	4
16	6	2	4	1
	8	0	2	1
		6	8	1
	9	4	4	2
	10	6	6	3
	12	8	4	3
	14	12	8	7

Forced strong regularity

Proposition (EGP)

Let Γ be a non-complete edge-regular graph with parameters (v, k, λ). Then Γ is strongly regular if:

(i) $v - 2k + \lambda \leq 1$, *or*

(ii) *there exists a strongly regular graph with parameters $(v, v - k - 1, 0, v - 2k + \lambda)$.*

Theorem (Greaves & Koolen, 2017)

Let Γ be a non-complete edge-regular graph with parameters (v, k, λ) having a regular 2-clique or 3-clique. Then Γ is strongly regular.

Reducing the table of parameters I

Excluding the cases covered in the previous results, we are left with...

v	k	λ	s	m
4	2	0	2	1
6	3	0	2	1
	4	2	3	2
8	4	0	2	1
	6	4	4	3
9	4	1	3	1
	6	3	3	2
10	5	0	2	1
	6	3	4	2
	8	6	5	4
12	5	2	4	1
	6	0	2	1
		4	6	1
	8	4	3	2
	9	6	4	3
	10	8	6	5

v	k	λ	s	m
14	7	0	2	1
	9	6	7	3
	12	10	7	6
15	6	1	3	1
		3	5	1
	8	4	5	2
	10	5	3	2
		6	5	3
	12	9	5	4
16	6	2	4	1
	8	0	2	1
		6	8	1
	9	4	4	2
	10	6	6	3
	12	8	4	3
	14	12	8	7

Reducing the table of parameters I

v	k	λ	s	m
12	5	2	4	1
	6	4	6	1
14	9	6	7	3
15	6	3	5	1
	8	4	5	2
16	6	2	4	1
	8	6	8	1
	9	4	4	2

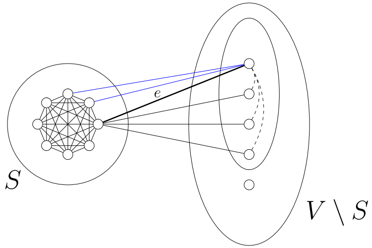
...only 8 parameter sets.

An inequality

Theorem (EGP)

Let Γ be an edge-regular graph with parameters (v, k, λ). Suppose Γ has an m-regular clique S of size $s \geq 2$. Then

$$k - \lambda - s + m - 1 \geq 0$$



Reducing the table of parameters II

Using this inequality, the table further reduces...

v	k	λ	s	m
12	5	2	4	1
	6	4	6	1
14	9	6	7	3
15	6	3	5	1
	8	4	5	2
16	6	2	4	1
	8	6	8	1
	9	4	4	2

Reducing the table of parameters II

v	k	λ	s	m
15	8	4	5	2
16	6	2	4	1
	9	4	4	2

...to only 3 parameter sets.

The equality case

Theorem (EGP)

Let Γ be an edge-regular graph with parameters (v, k, λ). Suppose Γ has an m-regular clique S of size $s \geq 2$ such that $k - \lambda - s + m - 1 = 0$. Then Γ is a complete graph or one of the following strongly regular graphs:

(i) *the Square Lattice graph $L_2(s)$;*

(ii) *the Triangular graph $T(s+1)$, where $s \geq 3$ $(V(T(n)) = \binom{[n]}{2}$, $AB \in E(T(n)) \iff |A \cap B| = 1)$;*

(iii) *the complete s-partite graph $K_{s \times 2}$, with parts of size 2.*

Reducing the table of parameters III

Finally, removing the sets satisfying the equality case...

v	k	λ	s	m
15	8	4	5	2
16	6	2	4	1
	9	4	4	2

Reducing the table of parameters III

v	k	λ	s	m
16	9	4	4	2

...we are left with only one possible parameter set!

Outline

A family of strongly regular graphs

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power. Let $Q(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}$ (the *hyperbolic quadric*).

Definition

The *affine polar graph* $V^+(2e, q)$ is the graph with vertices V, with two distinct vertices x, y being adjacent if and only if $Q(x - y) = 0$.

Lemma

The graph $V^+(2e, q)$ is a vertex transitive strongly regular graph.

Notation

$V^+(2e, q)$ is isomorphic to the graph defined on the set of all $(2 \times e)$-matrices over $\mathbb{F}_q$

$$\begin{pmatrix} x_1 & x_3 & \cdots & \cdots & x_{2e-1} \\ x_2 & x_4 & \cdots & \cdots & x_{2e} \end{pmatrix}$$

where two matrices are adjacent if and only if the scalar product of the first and the second rows of their difference is equal to 0.

The smallest case

Let $q = 2$. First we consider $e = 2$, and the cliques

$$W_1 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, W_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

Lemma

W_1, W_2 are 2-regular 4-cliques in $V^+(4, 2)$.

Edge switching

Definition

Let S, T be two disjoint subsets of vertices in a graph Γ. The *edge-switched* graph $\Gamma(S, T)$ is a graph defined on the same vertices as Γ. The edge uv is in $\Gamma(S, T)$ if $(u, v) \notin S \times T$ and uv is an edge in Γ, or $(u, v) \in S \times T$ and uv is not an edge in Γ.

Lemma

Let $\Gamma = V^+(4, 2)$ *and* $v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *Then* $\Gamma(W_i, v + W_i)$ *is strongly regular, with the same parameters as* $V^+(4, 2)$.

Doubling up

Notation

Let u be a vertex of $\Gamma = V^+(2e, 2)$ and S, T subsets of the vertices. We denote by $\Gamma(u, S, T)$ the result of applying an edge switching to Γ with respect to the sets $S, u + S$, and then switching with respect to $T, u + T$.

Lemma

Let $e = 2$ and v, W_1, W_2 as previously. Then $\Gamma(v, W_1, W_2)$ is a strictly Neumaier graph.

Extending

Now we consider general e. Define sets

$$W_1 = \left(\begin{array}{ccc|cc} * & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right), W_2 = \left(\begin{array}{ccc|cc} * & * & \cdots & * & 0 \\ 0 & 0 & \cdots & 0 & * \end{array}\right)$$

and let $v = \left(\begin{array}{ccc|cc} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right)$.

Lemma

W_1, W_2 are 2^{e-1}-regular 2^e-cliques in $V^+(2e, 2)$.

Theorem

$\Gamma(v, W_1, W_2)$ is a strictly Neumaier graph.

Outline

Summary

Results with Goryainov and Panasenko:

- The smallest strictly Neumaier graph, in $NG(16, 9, 4; 2, 4)$.
- The only possible parameters for a strictly Neumaier graph on $v \leq 24$ vertices are $(16, 9, 4; 2, 4)$ and $(24, 8, 2; 1, 4)$.
- The determination of all vertex-transitive strictly Neumaier graphs on $v \leq 47$ vertices (using classification by Holt and Royle).
- Two infinite sequences of strictly Neumaier graphs with parameters $(2^{2e}, (2^{e-1}+1)(2^e-1), 2(2^{e-2}+1)(2^{e-1}-1))$ containing 2^{e-1}-regular 2^e-cliques.

More Questions

- All known strictly Neumaier graphs contain regular cliques with nexus a power of 2. Are there Neumaier graphs with nexus not a power of 2?
- In all known strictly Neumaier graphs, the vertices can be partitioned into regular cliques. Are there strictly Neumaier graphs that cannot be partitioned into regular cliques?
- There are many prolific constructions of strongly regular graphs with regular cliques. Can we use these to find prolific constructions of strictly Neumaier graphs?

Thank you!