CI-property for decomposable Schur rings over an abelian group

Based on joint work with István Kovács

Grigory Ryabov

Novosibirsk State University

G2R2-2018,
Novosibirsk, August 06-19, 2018

S-rings

G is a finite group, e is the identity of G

A partition $\mathcal{S}$ of G is called a Schur partition if $\mathcal{S}$ satisfies the following properties:

- $\{e\} \in \mathcal{S}$,
- $X \in \mathcal{S} \;\Rightarrow\; X^{-1} \in \mathcal{S}$,
- for every $X, Y, Z \in \mathcal{S}$ the number $c_{X,Y}^{Z} = |Y \cap X^{-1}z|$ does not depend on $z \in Z$.

A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an S-ring (Schur ring) over G if there exists a Schur partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ such that $\mathcal{A} = Span_{\mathbb{Z}}\{\underline{X} : \; X \in \mathcal{S}\}$, where $\underline{X} = \sum_{x \in X} x$.

- The elements of $\mathcal{S}$ are called the basic sets of $\mathcal{A}$
- $\mathrm{rk}(\mathcal{A}) = |\mathcal{S}|$ is called the rank of $\mathcal{A}$

Schurian S-rings

- G is a finite group, e is the identity of G
- $G_{right} = \{x \mapsto xg,\ x \in G :\ g \in G\} \leq \mathrm{Sym}(G)$
- $\mathrm{Orb}(K, G)$ is the set of all orbits of $K \leq \mathrm{Sym}(G)$ on G

Theorem (Schur, 1933)

Let $K \leq \mathrm{Sym}(G)$ and $K \geq G_{right}$. Then $\mathrm{Orb}(K_e, G)$ is a Schur partition.

Schurian S-rings

- G is a finite group, e is the identity of G
- $G_{right} = \{x \mapsto xg,\ x \in G :\ g \in G\} \leq \mathrm{Sym}(G)$
- $\mathrm{Orb}(K, G)$ is the set of all orbits of $K \leq \mathrm{Sym}(G)$ on G

Theorem (Schur, 1933)

Let $K \leq \mathrm{Sym}(G)$ and $K \geq G_{right}$. Then $\mathrm{Orb}(K_e, G)$ is a Schur partition.

- An S-ring $\mathcal{A}$ over G is called schurian if $\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(K_e, G)$ for some $K \leq \mathrm{Sym}(G)$ such that $K \geq G_{right}$.
- There exist non-schurian S-rings. The first example of a non-schurian S-ring was found by Wielandt in 1964.
- A finite group G is called a Schur group if every S-ring over G is schurian (Pöschel, 1974).

Isomorphisms and automorphisms of S-rings

$\mathcal{A}$ and $\mathcal{A}^{'}$ are S-rings over groups G and $G^{'}$ respectively.

- A (combinatorial) isomorphism from $\mathcal{A}$ to $\mathcal{A}^{'}$ is defined to be a bijection $f: G \to G^{'}$ such that $\{\mathrm{Cay}(G,X): \ X \in \mathcal{S}(\mathcal{A})\}^f = \{\mathrm{Cay}(G^{'},X^{'}): \ X^{'} \in \mathcal{S}(\mathcal{A}^{'})\}$.
- $\mathrm{Iso}(\mathcal{A})$ is the set of all $f \in \mathrm{Sym}(G)$ such that f is an isomorphism from $\mathcal{A}$ to an S-ring over G.
- $\mathrm{Aut}(\mathcal{A}) = \bigcap\limits_{X \in \mathcal{S}(\mathcal{A})} \mathrm{Aut}(\mathrm{Cay}(G,X))$.

CI-S-rings

Definition (Hirasaka-Muzychuk, 2001)

An S-ring $\mathcal{A}$ over G is called a CI-S-ring if $\mathrm{Iso}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(G)$.

Proposition (Hirasaka-Muzychuk, 2001)

Let $\mathcal{A}$ be a schurian S-ring over G. Then the following conditions are equivalent:

- $\mathcal{A}$ is a CI-S-ring;
- Every two regular subgroups of $\mathrm{Aut}(\mathcal{A})$, which are isomorphic to G, are conjugate in $\mathrm{Aut}(\mathcal{A})$.

CI-S-rings

Definition (Hirasaka-Muzychuk, 2001)

An S-ring $\mathcal{A}$ over G is called a CI-S-ring if $\mathrm{Iso}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(G)$.

Proposition (Hirasaka-Muzychuk, 2001)

Let $\mathcal{A}$ be a schurian S-ring over G. Then the following conditions are equivalent:

- $\mathcal{A}$ is a CI-S-ring;
- Every two regular subgroups of $\mathrm{Aut}(\mathcal{A})$, which are isomorphic to G, are conjugate in $\mathrm{Aut}(\mathcal{A})$.

- If $\mathrm{rk}(\mathcal{A}) = 2$ then $\mathrm{Aut}(\mathcal{A}) = \mathrm{Sym}(G)$ and hence $\mathcal{A}$ is a CI-S-ring.
- If $\mathcal{A} = \mathbb{Z}G$ then $\mathrm{Aut}(\mathcal{A}) = G_{right}$ and hence $\mathcal{A}$ is a CI-S-ring.

CI-graphs and (D)CI-groups

- If $\sigma \in \mathrm{Aut}(G)$ then $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, S^\sigma)$.
- A Cayley graph $\mathrm{Cay}(G, S)$ is defined to be a CI-graph if $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ implies that $T = S^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$.
- A finite group G is defined to be a DCI-group (CI-group) if every (undirected) Cayley graph over G is a CI-graph.

CI-graphs and (D)CI-groups

- If $\sigma \in \mathrm{Aut}(G)$ then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,S^\sigma)$.
- A Cayley graph $\mathrm{Cay}(G,S)$ is defined to be a CI-graph if $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ implies that $T = S^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$.
- A finite group G is defined to be a DCI-group (CI-group) if every (undirected) Cayley graph over G is a CI-graph.

Problem (Babai)

Determine all DCI- and CI-groups.

- C.H. Li, On isomorphisms of finite Cayley graphs - survey, DM 256 (2002).
- C.H. Li, Z.P. Lu, P. Pálfy, Further restrictions on the struture of finite CI-groups, JACO 26 (2007).

Abelian DCI-groups

C_n is the cyclic group of order n.
$\mathcal{E}$ is the class of abelian groups whose every Sylow subgroup is elementary abelian.

- If G is ableian DCI-group then $G \in \mathcal{E}$ or Sylow 2-subgroup P of G is isomorphic to C_4 and $G/P \in \mathcal{E}$ (follows from the Li-Praeger-Xu's result).

Abelian DCI-groups

C_n is the cyclic group of order n.
$\mathcal{E}$ is the class of abelian groups whose every Sylow subgroup is elementary abelian.

- If G is ableian DCI-group then $G \in \mathcal{E}$ or Sylow 2-subgroup P of G is isomorphic to C_4 and $G/P \in \mathcal{E}$ (follows from the Li-Praeger-Xu's result).
- The following groups are DCI-groups:
  - C_n, C_{2n}, C_{4n}, where n is a square-free odd (Muzychuk);
  - C_p^e, where p is a prime and $e \leq 5$ (Elspas-Turner; Godsil; Alspah-Nowitz; Dobson; Hirasaka-Muzychuk, Morris, Feng-Kovács);
  - $C_p^2 \times C_q$, where p and q are distinct primes (Kovács-Muzychuk);
  - $C_p^3 \times C_q$, where p and q are distinct primes and $q > p^3$ (Somlai).

Abelian non-DCI-groups

- The following groups are non-DCI-groups:
  - C_2^e, where $e \geq 6$ (Nowitz);
  - C_3^e, where $e \geq 8$ (Spiga);
  - C_p^e, where $e \geq 2p+3$ (Somlai).

Abelian non-DCI-groups

- The following groups are non-DCI-groups:
  - C_2^e, where $e \geq 6$ (Nowitz);
  - C_3^e, where $e \geq 8$ (Spiga);
  - C_p^e, where $e \geq 2p + 3$ (Somlai).

Question

Is there a function $f(p)$ such that C_p^e is a DCI-group for $e < f(p)$ and a non-DCI-group for $e \geq f(p)$?

DCI-groups and S-rings

Proposition (Hirasaka-Muzychuk, 2001)

A finite group G is a DCI-group if and only if every schurian S-ring over G is a CI-S-ring.

DCI-groups and S-rings

Proposition (Hirasaka-Muzychuk, 2001)

A finite group G is a DCI-group if and only if every schurian S-ring over G is a CI-S-ring.

- To prove that given group G is a DCI-group it is sufficient to check that every schurian S-ring over G is a CI-S-ring.
- This approach was used to prove that the following groups are DCI-groups (here p and q are distinct primes):
  - C_p^4 (Hirasaka-Muzychuk,2001);
  - $C_p^2 \times C_q$ (Kovács-Muzychuk, 2009);
  - C_p^5 (Feng-Kovács, 2017).
- One of the main difficulties in this approach is to check that every decomposable schurian S-ring over given group is a CI-S-ring.

Decomposable S-rings

G is a finite group and $\mathcal{A}$ is an S-ring over G

- A subgroup $H \leq G$ is an $\mathcal{A}$-subgroup if $\underline{H} \in \mathcal{A}$.
- Let $L \trianglelefteq U \leq G$. A section U/L is an $\mathcal{A}$-section if U and L are $\mathcal{A}$-subgroups.
- If U/L is an $\mathcal{A}$-section then the module $\mathcal{A}_{U/L} = Span_{\mathbb{Z}} \{\underline{X}^{\pi} : \ X \in \mathcal{S}(\mathcal{A}), \ X \subseteq U\}$, where $\pi : U \to U/L$ is the canonical epimorphism, is an S-ring over U/L.

Definition (Evdokimov-Ponomarenko, 2001)

Let U/L be an $\mathcal{A}$-section. The S-ring $\mathcal{A}$ is called the U/L-wreath product or the generalized wreath product of $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ if $L \trianglelefteq G$ and every basic set of $\mathcal{A}$ outside U is a union of L-cosets.

- The U/L-wreath product is called nontrivial if $e \neq L$ and $U \neq G$.
- The S-ring $\mathcal{A}$ is said to be decomposable if $\mathcal{A}$ is the nontrivial U/L-wreath product for some $\mathcal{A}$-section U/L.

CI-porperty for decomposable S-rings

- In general case the generalize wreath product of two CI-S-rings can be non-CI-S-ring.

Example

Let $G = C_8$ and $L \leq U \leq G$ with $|L| = 2$ and $|U| = 4$. Then $\mathbb{Z}U$ and $\mathbb{Z}(G/L)$ are CI-S-rings, however the U/L-wreath product of $\mathbb{Z}U$ and $\mathbb{Z}(G/L)$ is not CI-S-ring.

CI-porperty for decomposable S-rings

- In general case the generalize wreath product of two CI-S-rings can be non-CI-S-ring.

Example

Let $G = C_8$ and $L \leq U \leq G$ with $|L| = 2$ and $|U| = 4$. Then $\mathbb{Z}U$ and $\mathbb{Z}(G/L)$ are CI-S-rings, however the U/L-wreath product of $\mathbb{Z}U$ and $\mathbb{Z}(G/L)$ is not CI-S-ring.

Question

- When the generalized wreath product of two CI-S-rings is a CI-S-ring?
- When the generalized wreath product of two CI-S-rings over a group from $\mathcal{E}$ is a CI-S-ring?

Main result

- If $\mathcal{A}$ is an S-ring over G then put $\mathrm{Aut}_G(\mathcal{A}) = \mathrm{Aut}(\mathcal{A}) \cap \mathrm{Aut}(G)$.
- For a set $\Delta \subseteq \mathrm{Sym}(G)$ and a section S of G we set $\Delta^S = \{f^S : \ f \in \Delta, \ S^f = S\}$.

Theorem (Kovács-R., 2018)

Let $G \in \mathcal{E}$, $\mathcal{A}$ an S-ring over G, and U/L an $\mathcal{A}$-section. Suppose that $\mathcal{A}$ is the nontrivial U/L-wreath product and the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are CI-S-rings. Then $\mathcal{A}$ is a CI-S-ring whenever

$$\mathrm{Aut}_{U/L}(\mathcal{A}_{U/L}) = \mathrm{Aut}_U(\mathcal{A}_U)^{U/L}\,\mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^{U/L}.$$

In particular, $\mathcal{A}$ is a CI-S-ring if $\mathrm{Aut}_{U/L}(\mathcal{A}_{U/L}) = \mathrm{Aut}_U(\mathcal{A}_U)^{U/L}$ or $\mathrm{Aut}_{U/L}(\mathcal{A}_{U/L}) = \mathrm{Aut}_{G/L}(\mathcal{A}_{G/L})^{U/L}$.

Corollary of Theorem

$\mathcal{A}$ is an S-ring over a group G

- $\mathcal{A}$ is called cyclotomic if $\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(K, G)$ for some $K \leq \mathrm{Aut}(G)$.
- $K_1, K_2 \leq \mathrm{Sym}(G)$ are 2-equivalent if $\mathrm{Orb}(K_1, G^2) = \mathrm{Orb}(K_2, G^2)$. In this case we write $K_1 \approx_2 K_2$.
- $\mathcal{A}$ is 2-minimal if $\{K \leq \mathrm{Sym}(G) : \ K \geq G_{right} \text{ and } K \approx_2 \mathrm{Aut}(\mathcal{A})\} = \{\mathrm{Aut}(\mathcal{A})\}$.
- $K_1, K_2 \leq \mathrm{Aut}(G)$ are Cayley equivalent if $\mathrm{Orb}(K_1, G) = \mathrm{Orb}(K_2, G)$. In this case we write $K_1 \approx_{Cay} K_2$.
- $\mathcal{A}$ is Cayley minimal if $\mathcal{A}$ is cyclotomic and $\{K \leq \mathrm{Aut}(G) : \ K \approx_{Cay} \mathrm{Aut}_G(\mathcal{A})\} = \{\mathrm{Aut}_G(\mathcal{A})\}$.
- $\mathbb{Z}G$ is 2- and Cayley minimal.

Corollary of Theorem

$\mathcal{A}$ is an S-ring over a group G

- $\mathcal{A}$ is called cyclotomic if $\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(K, G)$ for some $K \leq \mathrm{Aut}(G)$.
- $K_1, K_2 \leq \mathrm{Sym}(G)$ are 2-equivalent if $\mathrm{Orb}(K_1, G^2) = \mathrm{Orb}(K_2, G^2)$. In this case we write $K_1 \approx_2 K_2$.
- $\mathcal{A}$ is 2-minimal if $\{K \leq \mathrm{Sym}(G) : \ K \geq G_{right} \text{ and } K \approx_2 \mathrm{Aut}(\mathcal{A})\} = \{\mathrm{Aut}(\mathcal{A})\}$.
- $K_1, K_2 \leq \mathrm{Aut}(G)$ are Cayley equivalent if $\mathrm{Orb}(K_1, G) = \mathrm{Orb}(K_2, G)$. In this case we write $K_1 \approx_{Cay} K_2$.
- $\mathcal{A}$ is Cayley minimal if $\mathcal{A}$ is cyclotomic and $\{K \leq \mathrm{Aut}(G) : \ K \approx_{Cay} \mathrm{Aut}_G(\mathcal{A})\} = \{\mathrm{Aut}_G(\mathcal{A})\}$.
- $\mathbb{Z}G$ is 2- and Cayley minimal.

Corollary

Under assumption of Theorem suppose that $\mathcal{A}$ is cyclotomic and $\mathcal{A}_{U/L}$ is 2-minimal or Cayley minimal. Then $\mathcal{A}$ is a CI-S-ring.

Application of Theorem to decomposable S-rings over an elementary abelian group

$G = C_p^e$, where p is a prime and $e \geq 1$

- An S-ring $\mathcal{A}$ over G is called a p-S-ring if $|X| = p^k$ for every $X \in \mathcal{S}(\mathcal{A})$.
- To prove that G is a DCI-group it is sufficient to prove that every cyclotomic p-S-ring over G is a CI-S-ring (follows from Kovács-Feng's result).

Application of Theorem to decomposable S-rings over an elementary abelian group

- The proof that every decomposable p-S-ring over C_p^e, where $e \leq 4$, is a CI-S-ring not using Theorem takes approximately 5 pages.
- The proof that every decomposable p-S-ring over C_p^e, where $e \leq 4$, is a CI-S-ring using Theorem takes few lines.
- The proof that every decomposable cyclotomic p-S-ring over C_p^5 is a CI-S-ring not using Theorem takes approximately 9 pages.
- The proof that every decomposable p-S-ring over C_p^e, where $e \leq 4$, is a CI-S-ring using Theorem takes 1 page.
- Using Theorem it is possible to prove that in most cases decomposable cyclotomic p-S-ring over C_p^6 is a CI-S-ring.

Application of Theorem to decomposable S-rings over an elementary abelian group

- The proof that every decomposable p-S-ring over C_p^e, where $e \leq 4$, is a CI-S-ring not using Theorem takes approximately 5 pages.
- The proof that every decomposable p-S-ring over C_p^e, where $e \leq 4$, is a CI-S-ring using Theorem takes few lines.
- The proof that every decomposable cyclotomic p-S-ring over C_p^5 is a CI-S-ring not using Theorem takes approximately 9 pages.
- The proof that every decomposable p-S-ring over C_p^e, where $e \leq 4$, is a CI-S-ring using Theorem takes 1 page.
- Using Theorem it is possible to prove that in most cases decomposable cyclotomic p-S-ring over C_p^6 is a CI-S-ring.

Question

Let p be an odd prime. Is C_p^6 a DCI-group?