On Edge-transitive Factorizations of Complete Uniform Hypergraphs

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This is joint work with Zaiping Lu

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Outline

1. Edge-transitive factorizations of complete uniform hypergraphs
   - Definitions and background
   - Edge-transitive homogeneous factorizations of $\mathcal{K}_n^k$
   - Symmetric factorizations of $\mathcal{K}_n^k$
2. Cayley hypergraph
   - Definitions and background
   - Normality of Cayley hypergraphs
   - CHI-property of Cayley hypergraphs

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Theorem 1.1 (Baranyai's Theorem, 1975)

If n is divisible by k then the complete k-hypergraph $\mathcal{K}_n^k$ admits a 1-factorization of index $\binom{n-1}{k-1}$.

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- A family of trivial examples:
  $V = \{1, 2, 3, \cdots, 2k\}$
  $\mathcal{U}_{2k}^{k} := \{\{e, [2k] \setminus e\} | e \in \binom{[2k]}{k}\}$
  $\mathcal{U}_{2k}^{k}$ is S_{2k}-invariant, that is, $\{e, e'\}^{\sigma} \in \mathcal{U}_{2k}^{k}$ for all $\sigma \in S_{2k}$ with $\{e, e'\} \in \mathcal{U}_{2k}^{k}$; $(S_{2k})_{\{e,e'\}}$ is transitive on V.

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  - Edge-transitive homogeneous factorizations i.e. $\mathsf{SHF}(n, k, s)$

Known results

1. Cameron and Korchmaros (1993): One-factorizations of complete graphs with a doubly transitive automorphism group.
2. Li and Praeger (2003): HF for complete graphs.
3. Sibley (2004): SF for complete graph with Aut$\mathcal{F}$ is not affine permutation group but acts 2-transitive on V.
4. Li, Lim and Praeger (2009): HF of complete graphs with all factors admitting a common edge-transitive group.

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Remark 1.3 (Chen and Lu)

SF *for complete graph with* $\mathsf{Aut}\mathcal{F}$ *an affine 2-homogeneous permutation group is* SHF.

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- Classify edge-transitive homogeneous factorizations of complete k-hypergraphs, where $k \geq 3$.

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- Classify symmetric factorizations of complete k-hypergraphs, where $k \geq 3$.

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- Aut$\mathcal{F}$ is a k-homogeneous permutation groups on V
- $\mathsf{F}(n, k, s)$ is the immprimitive block systems of Aut$\mathcal{F}$ acting on $V^{\{k\}}$

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Construction of SHF

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- $E = e^{MH}$ consists of $|MH : (MG_e)|$ orbits of M on $V^{\{k\}}$
- $\mathcal{F} := \{E^g \mid g \in G\}$
  edge-transitive homogeneous (k, s)-factorizations on V, where $s = |G : (MH)|$

Theorem 1.4 (Chen and Lu,2018)

There exists an SHF(n,k,s) *for* $n \geq 2k \geq 6$ *and* $s \geq 2 \Leftrightarrow (n,k,s)$ *is one of* $(32,3,5)$, $(32,3,31)$, $(33,4,5)$, $(2^d,3,\frac{(2^d-1)(2^{d-1}-1)}{3})$ *and* $(q+1,3,2)$, *where* $d \geq 3$ *and* q *a prime power with* $q \equiv 1 (mod\ 4)$. *In particular, there is no* SHF 1*-factorization of index* $s \geq 2$ *and order* $n \geq 6$.

Remark

A k-hypergraph on n vertices is t-subset regular if there is a constant $\lambda \geq 1$ such that each t-subset of V is contained in exactly λ edges. Note that each t-subset regular k-hypergraph is a $t-(v,k,\lambda)$ design. The factors of $\mathcal{F}$ in Theorem 1.4 are t-subset regular k-hypergraph with t and λ listed in Table 1

n	k	s, N	t	λ	Condition
32	3	5	2	6	
32	3	31	1	15	
33	4	5	3	6	
2^d	3	$\frac{(2^d-1)(2^{d-1}-1)}{3}$	1	3	$d \geq 3$
$q+1$	3	2	2	$\frac{q-1}{2}$	$q \equiv 1 \pmod 4$

Table: The parameters t and λ.

Remark

A large set of $t-(v,k,\lambda)$ designs of size N, denoted by $LS[N](t,k,n)$, is a partition of $V^{\{k\}}$ $(|V|=n)$into block sets of N disjoint $t-(v,k,\lambda)$ designs. Clearly, each $\mathcal{F}$ in Theorem 1.4 is an $LS[N](t,k,n)$ in which all designs are flag-transitive and admit a common point-transitive group, where N,k,t and n are listed in Table 1.

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SF(n, k, s)

Clearly, SHF is SF. Then we want to classify all the SF(n, k, s) with $k \geq 3$. By the way, we obtain all the symmetrical 1-factorizations with $k \geq 3$.

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- $\mathcal{F} := \{\{e^{Xg}\} \mid g \in G\}$
  symmetric (k, s)-factorizations on V, where $s = |G : X|$

Theorem 1.5 (Chen and Lu, 2017)

For $6 \leq 2k \leq n$, there is a symmetric 1-factorization $\Leftrightarrow$ either $n = 2k$ or $(n,k) \in \{(q+1,3),(24,4)\}$, q is a prime power with $11 \leq q \equiv 2 (mod\ 3)$.

Theorem 1.6 (Chen and Lu, 2017)

Let $\mathcal{F}$ be an $SF(n,k,s)$ with $n \geq 2k \geq 6$ and $s \geq 2$. Then

1. *$\mathcal{F}$ is homogeneous; or*
2. *$\mathcal{F}$ is a 1-factorization; or*
3. $(n,k,s) \in \{(8,3,7),(12,3,11),(20,3,57),(12,5,66)\}$

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Definition 2.1

Given a finite group G and a symmetric connector set $S \subset G \setminus e$, the Cayley graph, denoted $\mathsf{Cay}(G, S)$, is the graph with $V = G$ and $E = \{(x, y) \in V \times V \mid yx^{-1} \in S\}$ (i.e $y = sx$ for some $s \in S$.)

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Definition 2.2 (M. Buratti, 1994)

Let G be a finite group, Ω a subset of $G \setminus e$ and t an integer satisfying $2 \leq t \leq max\{o(\omega) \mid \omega \in \Omega\}$; the t-Cayley hypergraph $\mathcal{H} = t - \mathsf{Cay}(G, \Omega)$ of G over Ω is defined by:

- $V(\mathcal{H}) := G$;
- $E(\mathcal{H}) := \{\{g, \omega g, \cdots, \omega^{t-1} g\} \mid g \in G, \omega \in \Omega\}$.

Definition 2.3 (J. Lee and Y. S. Kwon, 2013)

Let G be a finite group and let $\mathcal{S}$ be a set of subsets $S_1, S_2, S_3, \cdots, S_k$ of $G \setminus \{e\}$. A Cayley hypergraph $\mathsf{CH}(G, \mathcal{S})$ *is defined by:*

- $V(\mathsf{CH}) = G$;
- $E(\mathsf{CH}) = \{g, S_i g | g \in G, S_i \in \mathcal{S}\}$.

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- A Cayley hypergraph $\mathsf{CH}(G, \mathcal{S})$ is said to be a normal Cayley hypergraph if $R(G) \trianglelefteq \mathsf{Aut}(\mathsf{CH}(G, \mathcal{S}))$. (M. Alaeiyan, 2007)

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- A Cayley hypergraph $\mathsf{CH}(G, \mathcal{S})$ is called a CHI-hypergraph of G, if whenever $\mathsf{CH}(G, \mathcal{S}) \cong \mathsf{CH}(G, \mathcal{T})$, there is an element $\sigma \in \mathsf{Aut}(G)$ such that $\mathcal{S}^\sigma = \mathcal{T}$, and $\mathcal{S}$ is called a CHI-subset.

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- Babai and Cameron (2015) Except for the alternating groups and finitely many others, every primitive permutation group is the full automorphism group of an edge-transitive hypergraph
- Spiga (2016) Obtain the explicit list of finite primitive groups which are not automorphism groups of edge-transitive hypergraphs

Theorem 2.1 (Chen and Lu)

Let $\mathcal{H} = \mathsf{CH}(G, \mathcal{S})$ *be a Cayley hypergraph. Then* $\mathcal{H}$ *is connected if and only if* $\cup_{i=1}^{k} S_i$ *generate* G *where* $\mathcal{S} = \{S_1, S_2, \cdots, S_k\}$.

Theorem 2.2 (Chen and Lu)

Let $\mathsf{CH}(G, \mathcal{S})$ *be a Cayley hypergraph. Then the degree of this hypergraph is* $d_{CH} \leq k + |S_1| + |S_2| + |S_3| + \cdots + |S_k|$.

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Theorem 2.3 (Chen and Lu)

Let $S_i \neq \{1, S_x \setminus s_{xy}\}^{R(a)\sigma}$ *for any* $a \in G$, $\sigma \in \mathsf{Aut}(G)$ *where* $x \in \{1, 2, 3, \cdots, k\}$ *and* $s_{xy} \in S_x$. *Then the Cayley hypergraph* $\mathsf{CH}(G, \mathcal{S})$ *is normal if and only if* $\mathsf{A}_1 \subset \mathsf{Aut}(G)$.

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Theorem 2.4 (Chen and Lu)

Let $\mathcal{H}$ *be a vertex-transitive hypergraph whose automorphism group* $\mathsf{Aut}(\mathcal{H})$ *is abelian. Then* $\mathsf{Aut}(\mathcal{H})$ *acts regularly on the vertex set of* $\mathcal{H}$ *and* $\mathcal{H}$ *is a Cayley hypergraph* $\mathsf{CH}(G, \mathcal{S})$. *Moreover if* $\mathcal{S}^{-1} = \mathcal{S}$, *then* $\mathsf{Aut}(\mathsf{CH}(G, \mathcal{S})) \cong \mathbb{Z}_2^k$ *for some integer* k.

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Theorem 2.5 (Chen and Lu)

Let $X := \mathsf{CH}(G, \mathcal{S})$ *and* $\mathsf{A} := \mathsf{Aut}(X)$. *Then* $\mathcal{S}$ *is a* CHI*-subset of* G *if and only if for any* $\sigma \in \mathrm{Sym}(G)$ *whenever* $\sigma R(G)\sigma^{-1} \leq \mathsf{A}$ *there is an element* $a \in \mathsf{A}$ *such that* $aR(G)a^{-1} = \sigma R(G)\sigma^{-1}$.

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Theorem 2.6 (Chen and Lu)

Let G *be a finite* p*-group for some prime* p. *If* $X = \mathsf{CH}(G, S)$ *with* $\mathcal{S} = \{S_1, S_2, S_3, \cdots, S_k\}$ *and* $\Sigma_{i=1}^{k}|S_i| + k < p$, *then* X *is a* CHI*-hypergraph.*

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Thank you for your attention!