Structure and Automorphism group of Involution $G-$Graphs and Cayley graphs

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$G-$graph

Let G be a finite group and $S = \{s_1, s_2, \cdots, s_k\}$ a nonempty set of elements of G, $k \geq 1$. A pair (G, S) is called a $S-$group and G is a $S-$group if $G = \langle S \rangle$.

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$\forall s \in S$, $G = \sqcup_{x \in T_s} \langle s \rangle x$, where T_s is a write transversal of $\langle s \rangle$
Let $g_s : G \to G$, $g_s(x) = sx$ of S_G and for $x \in G$, let us consider the cycles:

$$\mathbf{(s)x = (x, sx, s^2x, \cdots, s^{o(s)-1}x)}$$

$G-$graph [Bretto (2005)]

- $V(\Phi(G,S)) =$ The distinct cycles in the decomposition of g_s, $s \in S$, i.e., $V = \sqcup_{s \in S} V_s$ with $V_s = \{(s)x, x \in T_s\}$.

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- For each $(s)x$, $(t)y \in V$, if $|\langle s \rangle x \cap \langle t \rangle y| = d$, $d \geq 1$ then $\{(s)x, (t)y\}$ is a d-edge.

$G-$graph properties

$\Phi(G, S)$ is $|S| = k-$partite graph and every vertex has $o(s)-$loops.
It has no multi edges iff for all $s, t \in S$, $\langle s \rangle \cap \langle t \rangle = \{1\}$.
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$\tilde{\Phi}(G,S)$ is connected $\Leftrightarrow G=\langle S\rangle$.

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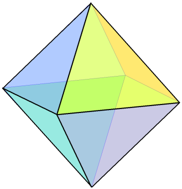
- $K_{2,n} = \tilde{\Phi}(D_{2n}, \{a,b\})$,

- The octahedral graph $= \tilde{\Phi}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1,0),(0,1),(1,1)\})$



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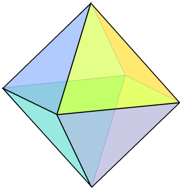

- The cuboctahedral graph
$= \tilde{\Phi}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \{(1,0,0),(0,1,0),(0,0,1)\})$

The hypercube graph Q_n is an $n-$regular graph with $|V| = 2^n$ and $|E| = 2^{n-1}n$. The vertices are all $n-$dimensional vectors on $\{0, 1\}$. Two vectors are adjacent iff they differ in a single element.

- $Q_3 = \tilde{\Phi}(A_4, \{(1, 2, 3), (1, 3, 4)\})$

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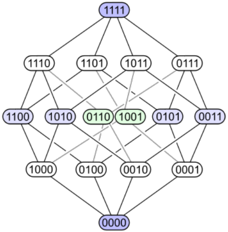
- $Q_3 = \tilde{\Phi}(A_4, \{(1,2,3),(1,3,4)\})$
- $Q_4 = \tilde{\Phi}(G = SmallGroup(32,6), \{f1, f1 * f2\})$, $G = ((C_4 \times C_2) : C_2) : C_2$



Möbius ladder

The Möbius ladder M_n = A cubic circulant graph with an even n of vertices.

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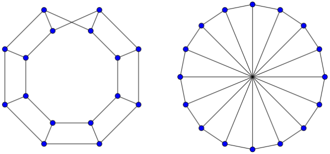

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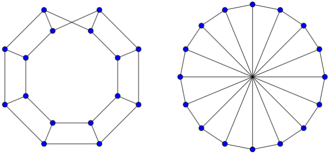

M_n is a bipartite $G-$graph $\tilde{\Phi}(G, S)$ iff $G = \langle S \rangle$, $S = \{s, t\}$ such that $o(s) = o(t) = 3$ and $|E| = |G| \equiv \pm 1 (mod 4)$.

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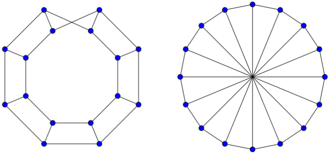

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M_{50}

```
gap> G:=SmallGroup(75,2);
<pc group of size 72 with 5 generators>
gap> S:=FindGenerators(G,[3,3]);
[ f1, f1*f2 ]
gap> graph:=GGraph(G,S);
```

$G-$graph morphism

For a given $G-$graph $\widetilde{\Phi} = \tilde{\Phi}(G, S) = (V, E)$ and any $g \in G$, we associate the map $\delta : G \rightarrow Aut(\widetilde{\Phi})$, $\delta(g) = (\delta_{g^{-1}}, \bar{\delta}_{g^{-1}})$

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- $\bar{\delta}_{g^{-1}} : E \rightarrow E$,
  $\bar{\delta}_{g^{-1}}([(s)x, (t)y], u) = ([(s)xg^{-1}, (t)yg^{-1}], ug^{-1})$

$\delta(G)$ acts transitively on every V_s, $s \in S$ and all $(s)x$, $x \in T_s$ are of the same order. .

$G-$graph and Cayley graph

Bretto et.,al. 2008

Let $G = \langle S \rangle$ and $\tilde{\Phi} = \tilde{\Phi}(G, S)$:

1) If $S = \{\alpha, \beta\}$ and $A = (\langle \alpha \rangle \cup \langle \beta \rangle) \setminus \{1\}$, Then $L(\tilde{\Phi}) \simeq Cay(G, A)$.

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1) If $S = \{\alpha, \beta\}$ and $A = (\langle \alpha \rangle \cup \langle \beta \rangle) \setminus \{1\}$, Then $L(\tilde{\Phi}) \simeq Cay(G, A)$.
2) If $\forall s \in S$, $o(s) = m > 0$ and if
   i) $\exists A \leq Aut_S(G)$ which acts regularly on S,
   ii) $\exists B \leq G$ of size $|B| = \frac{|G|}{m}$ such that $\forall f \in A, f(B) = B$ and $B \cap \langle s \rangle = \{1\}$, for all $s \in S$.

   Then $H = \delta(B) \rtimes A \leq Aut(\tilde{\Phi})$ acts regularly on $V(\tilde{\Phi})$ and so $\tilde{\Phi} \cong Cay(H, T)$, for $T \subseteq H$.

$$
\begin{aligned}
D_{2n} &= \langle a, b | a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\
V_{8n} &= \langle a, b | a^{2n} = b^4 = 1, aba = b^{-1}, ab^{-1}a = b \rangle, \\
SD_{8n} &= \langle a, b | a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle, \\
T_{4n} &= \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, \\
U_{2nm} &= \langle a, b | a^{2n} = b^m = 1, aba^{-1} = b^{-1} \rangle.
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\widetilde{\Phi}(D_{2n}, \{a, b\}) \cong K_{2,n}, \quad \widetilde{\Phi}(U_{6n}, \{a, b\}) \cong K_{3,2n},
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\widetilde{\Phi}(V_{8n}, \{a, b\}) \cong K_{4,2n}, \quad \widetilde{\Phi}(SD_{8n}, \{a, b\}) \cong K_{2,4n},
$$

$$
\widehat{\Phi}(T_{4n}, \{a, b\}) \cong K_{2,n}.
$$

Table : $G = A_n$, $S = \{a, b\}$, $\widehat{\Phi} = \widehat{\Phi}(A_n, S)$ and $\Gamma = Cay(A_n, S)$ for $n = 4, 5, 6, 7, 8$

G	A_4	A_5	A_6	A_7	A_8
$Aut(G)$	S_4	S_5	$(A_6 : \mathbb{Z}_2) : \mathbb{Z}_2$	S_7	S_8
$Aut(\widehat{\Phi})$	$\mathbb{Z}_2 \times S_4$	$\mathbb{Z}_2 \times A_5$	$\mathbb{Z}_2 \times A_6$	S_7	S_8
$Aut(\Gamma)$	$\mathbb{Z}_2 \times S_4$	$\mathbb{Z}_2 \times A_5$	$\mathbb{Z}_2 \times A_6$	S_7	S_8

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Conjecture

For $n \geq 7$,

$$Aut(A_n) \cong Aut(\widehat{\Phi}(A_n, S)) \cong Aut(Cay(A_n, S)).$$

Table : $G = L_2(p), A = Aut(\widehat{\Phi}(G, S = \{a, b\}))$ for $p = 2, 3, 5, 7, 11, 13, 17, 19, 23$

G	$L_2(2)$	$L_2(3)$	$L_2(5)$	$L_2(7)$	$L_2(11)$
A	D_{12}	S_4	$\mathbb{Z}_2 \times A_5$	$L_2(7) : \mathbb{Z}_2$	$L_2(11) : \mathbb{Z}_2$
G	$L_2(13)$	$L_2(17)$	$L_2(19)$	$L_2(23)$	
A	$L_2(13) \times \mathbb{Z}_2$	$L_2(17) \times \mathbb{Z}_2$	$L_2(19) : \mathbb{Z}_2$	$L_2(23) : \mathbb{Z}_2$	

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For $p = 7$, $Aut(\widehat{\Phi}(G, S)) \cong \mathbb{Z}_2 \times Aut(G)$ and $\widehat{\Phi}(L_2(7), S)$ is a $3-$regular connected $G-$graph.

$G-$graph $\widehat{\Phi}(L_2(p), \{a, b\})$

Proposition

Let $p \geq 3$ be a prime number and $G = L_2(p) = \langle a, b \rangle$ (standard generators).
$\widehat{\Phi}(G, \{a, b\})$ is a bipartite, semi-regular connected graph with two parts V_a and V_b.
Each vertex of V_a and V_b is of degree 3 and $\frac{p-1}{2}$, respectively.

$G = L_2(p)$, $\widehat{\Phi} = \widehat{\Phi}(G, Inv)$, $\Gamma = Cay(G, Inv)$, where $p = 2, 3, 5, 7$

G	$L_2(2)$	$L_2(3)$	$L_2(5)$
$\langle Inv \rangle$	S_3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	A_5
$Aut(G)$	S_3	S_4	S_5
$Aut(\widehat{\Phi})$	$S_3 \wr \mathbb{Z}_2$	$((((A_4 \wr \mathbb{Z}_2) \times A_4) : \mathbb{Z}_2) : \mathbb{Z}_3) : \mathbb{Z}_2) : \mathbb{Z}_2$	$(A_5 \wr \mathbb{Z}_2) : \mathbb{Z}_2$
$Aut(\Gamma)$	$S_3 \wr \mathbb{Z}_2$		$(A_5 \wr \mathbb{Z}_2) : \mathbb{Z}_2$

G	$L_2(7)$	$L_2(11)$
$\langle Inv \rangle$	$L_2(7)$	$L_2(11)$
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$Aut(\widehat{\Phi})$	$(L_2(7) \wr \mathbb{Z}_2) : \mathbb{Z}_2$	$(L_2(11) \wr \mathbb{Z}_2) : \mathbb{Z}_2$
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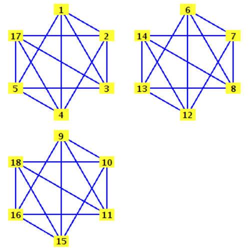
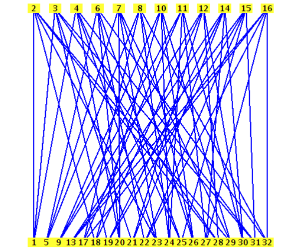
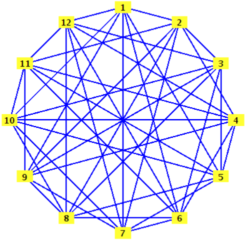


$\widehat{\Phi}(L_2(3), Inv)$



$\widehat{\Phi}(A_5, \{(1,2,3,4,5),(3,4,5)\})$



$Cay(D_{12}, Inv)$

Let (G, S) be an $S-$group such that all elements of S are of the same order.
For any $f \in Aut(G)$, since f preserves the order of each element $s \in S$, then we can see that $f(S) = S$ and $f \in Aut_S(G)$.
Since $G \leq Aut(G)$ and every $s \in S$ has the same order, then $Aut(G) = Aut_S(G)$ and since $Aut_S(G) \leq Aut(\widehat{\Phi}(G, S))$ then

$$G \leq Aut(\widehat{\Phi}(G, S)).$$

Theorem

Let G be a finite simple group then
$G \cong Aut(\widetilde{\Phi}(G, Inv)) \iff$
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Proof.

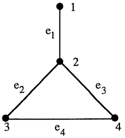
Suppose $G \cong Aut(\widetilde{\Phi}(G, Inv))$. Since $\frac{G}{Z(G)} \cong Inn(G)$ and G is simple, then $G \cong Inn(G)$. Also $Inn(G) \leqslant Aut(G) \leqslant Aut(\widetilde{\Phi})$ implies that

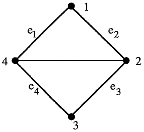
$$G \cong Inn(G) \cong Aut(G) \cong Aut(\widetilde{\Phi}(G, Inv))$$

and $Out(G) \cong \frac{Aut(G)}{Inn(G)} = 1$. This satisfies for the sporadic group G with $Out(G) = 1$. The converse is obviously true . □

Lemma

1. For any non-empty simple connected graph Γ,
$Aut(\Gamma) \cong Aut(L(\Gamma))$, *except* K_2, K_4 *and the followings graphs, which non of them are Cayley graphs.*





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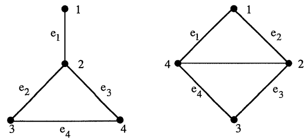

Lemma

2. Let (G, Inv) *be an* $Inv-$*group with* $|Inv| \geq 2$*. Then*

$$L(Cay(G, Inv)) \cong \widetilde{\Phi}(G, Inv).$$

Theorem

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It is the result of Lemma 1 and Lemma 2. □

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It is the result of Lemma 1 and Lemma 2. □

Lemma

For a finite group G, if $|Inv| = 1$ then

$$Aut(Cay(G, Inv)) \cong \mathbb{Z}_2 \wr S_{\frac{|G|}{2}},$$

$$Aut(\widehat{\Phi}(G, Inv)) \cong Aut(\overline{K_{|G|}}) \cong S_{|G|}$$

Lemma

i) $G = T_{4n}$,

$$Aut(G) \cong \mathbb{Z}_{2n} \rtimes \mathbb{Z}^{1}{}_{2n}$$

of order $2n\phi(2n)$, *where* $\mathbb{Z}^{1}{}_{2n}$ *is the group of units of* $\mathbb{Z}_{2n}$.

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ii) $G = V_{8n}$ *and* $n > 1$, *then* $Aut(G)$ *is of order* $4n\phi(2n)$ *and for* $n = 1$, $G = D_8$, *then* $Aut(G) \cong D_8$.

Lemma

iii) $G = U_{2nm}$, if $2 \mid m$ or $2|n$, then $|Aut(U_{2nm})| = m\phi(m)\phi(2n)$ and $Aut(U_{2nm}) = \{f_{i,j,r} | f_{i,j,r}(a) = a^i b^j, f_{i,j,r}(b) = b^r, (i, 2n) = (r, m) = 1\}$

$$= \mathbb{Z}^*{}_{2n} \times (\mathbb{Z}_m \rtimes \mathbb{Z}^*{}_m)$$

where $\mathbb{Z}^*{}_{2n}$ is the group of invertible elements of $\mathbb{Z}_{2n}$.

Lemma

iii) $G = U_{2nm}$, *if* $2 \mid m$ *or* $2|n$, *then* $|Aut(U_{2nm})| = m\phi(m)\phi(2n)$ *and* $Aut(U_{2nm}) = \{f_{i,j,r} | f_{i,j,r}(a) = a^i b^j, f_{i,j,r}(b) = b^r, (i, 2n) = (r, m) = 1\}$

$$= \mathbb{Z}^*{}_{2n} \times (\mathbb{Z}_m \rtimes \mathbb{Z}^*{}_m)$$

where $\mathbb{Z}^*{}_{2n}$ *is the group of invertible elements of* $\mathbb{Z}_{2n}$.

If $2 \nmid m$ *and* $2 \nmid n$, *then* $|Aut(U_{2nm})| = 2m\phi(m)\phi(2n)$ *and*

$$Aut(U_{2nm}) \cong (\mathbb{Z}^*{}_{2n} \times (\mathbb{Z}_m \rtimes \mathbb{Z}^*{}_m)) \rtimes \mathbb{Z}_2$$

Theorem

1) $Aut(Cay(T_{4n}, Inv)) = \mathbb{Z}_2 \wr S_n, \quad Aut(\widehat{\Phi}(T_{4n}, Inv)) \cong S_{2n}.$

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2) $2 \nmid n$, $Cay(D_{2n}, Inv) \cong K_{n,n}$,

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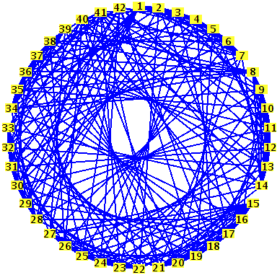
2) $2 \nmid n$, $Cay(D_{2n}, Inv) \cong K_{n,n}$,

$$Aut(Cay(D_{2n}, Inv)) \cong Aut(\widehat{\Phi}(D_{2n}, Inv)) \cong S_n \wr \mathbb{Z}_2.$$

$2|n$, *$Cay(D_{2n}, Inv)$ is a connected $(n+1)-$regular graph with $2n$ vertices and $\widehat{\Phi}(D_{2n}, Inv)$ is a connected $(n+1)-$partite graph of size $n(n+1)$.*



3) $G = SD_{8n}$, $2 \nmid n$,

$$Aut(Cay(SD_{8n}, Inv)) \cong Aut(\widehat{\Phi}(SD_{8n}, Inv)) \cong (D_8 \wr \mathbb{Z}_2) \wr S_n$$

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$$Inv = Inv(G) = \{b, a^{2n}b, a^{2n}, a^{3n}b, a^{n}b\},$$

$\langle Inv \rangle \cong SD_8 = D_8$ and $Cay(D_8, Inv)$ is a connected, $5-$regular graph with 8 vertices.

$$Cay(SD_{8n}, Inv) = nCay(SD_8, Inv)$$

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$$Cay(SD_{8n}, Inv) = nCay(SD_8, Inv)$$

Also $\widehat{\Phi}\ (G, Inv)$ is a disconnected graph which contains n copies of $\widehat{\Phi}\ (SD_8, Inv)$ that is a $8-$regular graph with 20 vertices.

SD_{8n}, $2|n$,

$$Aut(Cay(SD_{8n}, Inv)) \cong Aut(\widehat{\Phi}(SD_{8n}, Inv)) \cong S_4 \wr S_{2n}.$$

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$$Inv = Inv(G) = \{b, a^{2n}b, a^{2n}\}$$

and $\langle Inv \rangle \cong \mathbb{Z}_4$. Clearly $Cay(\mathbb{Z}_4, Inv) \cong K_4$ and $Cay(SD_{8n}, Inv) = 2nK_4$.

SD_{8n}, $2|n$,

$$Aut(Cay(SD_{8n}, Inv)) \cong Aut(\widehat{\Phi}(SD_{8n}, Inv)) \cong S_4 \wr S_{2n}.$$

$$Inv = Inv(G) = \{b, a^{2n}b, a^{2n}\}$$

and $\langle Inv \rangle \cong \mathbb{Z}_4$. Clearly $Cay(\mathbb{Z}_4, Inv) \cong K_4$ and $Cay(SD_{8n}, Inv) = 2nK_4$.

$\widehat{\Phi}(SD_{8n}, Inv) = 2n\widehat{\Phi}(\mathbb{Z}_4, Inv)$. and

$$Aut(Cay(SD_{8n}, Inv)) \cong Aut(Cay(\mathbb{Z}_4, Inv)) \wr S_{2n} \cong$$

$$Aut(\widehat{\Phi}(SD_{8n}, Inv)) \cong Aut(\widehat{\Phi}(\mathbb{Z}_4, Inv)) \wr S_{2n} \cong S_4 \wr S_{2n}$$

4) $G = V_{8n}$

$Inv(V_{8n}) = \{a^i b, a^i b^{-1} | 1 \leq i \leq 2n-1, 2 \nmid i\} \cup \{a^n, b^2, a^n b^2\}$.

$2 \nmid n$, $Z(V_{8n}) = \{1, b^2\}$ and

$$\langle Inv \rangle \cong (\mathbb{Z}_{2n} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$$

$\langle Inv \rangle = G$ and $Cay(G, Inv)$ is a connected $(2n+3)-$regular graph.

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$\langle Inv \rangle = G$ and $Cay(G, Inv)$ is a connected $(2n+3)-$regular graph.

$2 \mid n$, $Z(V_{8n}) = \{1, a^n, b^2, a^n b^2\}$,

$$n = 4k - 2 \Longrightarrow \langle Inv \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_n$$

$$n = 4k \Longrightarrow \langle Inv \rangle = \mathbb{Z}_2 \times D_{2n}$$

and $Cay(V_{8n}, Inv)$ is a disconnected graph with two isomorphic connected components.

$\widehat{\Phi}(V_{8n}, Inv)$

$\widehat{\Phi} = \widehat{\Phi}(V_{8n}, Inv)$ is a $4(n+1) - regular$ graph.
$2 \nmid n$, $\widehat{\Phi}$ is connected but when $2|n$ it has two isomorphic connected components.

5) $G = U_{2nm}$,
$2 \nmid m, n$:

$$Aut(Cay(U_{2nm}, Inv)) \cong (S_m \wr \mathbb{Z}_2) \wr S_n$$

$2 \nmid m$ and $2|n$

$$Aut(Cay(U_{2nm}, Inv)) \cong \mathbb{Z}_2 \wr S_{mn}.$$

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$2 \nmid m, n$:

$$Aut(Cay(U_{2nm}, Inv)) \cong (S_m \wr \mathbb{Z}_2) \wr S_n$$

$2 \nmid m$ and $2|n$

$$Aut(Cay(U_{2nm}, Inv)) \cong \mathbb{Z}_2 \wr S_{mn}.$$

$2|m$ and $2 \nmid n$,

$$Aut(Cay(U_{2nm}, Inv)) = Aut(Cay(U_{2m}, Inv)) \wr S_n$$

$2|m, n$,

$$Aut(Cay(U_{2nm}, Inv)) \cong S_4 \wr S_{\frac{mn}{2}}.$$

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Thank you!

Ferdossi

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