On vertex connectivity of Deza graphs

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August 16, 2016

Overview

Notations

We consider undirected graphs without loops and multiple edges.

For a graph Γ and its vertex x, define the neighbourhood of x:

$$\Gamma(x) := \{y \mid y \in V(\Gamma), y \sim x\}.$$

A graph Γ is called regular of valency k if

$$|\Gamma(x)| = k$$

holds for all $x \in \Gamma$.

The vertex connectivity of a graph Γ is the minimum number of vertices $\kappa(\Gamma)$ whose deletion from Γ disconnects it.

Deza graphs and Strongly regular graphs

A graph Δ is called a Deza graph with parameters (v, k, b, a) (usually $a < b$), if Δ has v vertices, and for any pair of vertices $x, y \in \Delta$:

$$|\Delta(x) \cap \Delta(y)| = \begin{cases} k, & \text{if } x = y, \\ a \text{ or } b, & \text{if } x \neq y. \end{cases}$$

$$\bigcup$$

A graph Γ is called strongly regular with parameters (v, k, λ, μ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$:

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k, & \text{if } x = y, \\ \lambda, & \text{if } x \sim y, \\ \mu, & \text{if } x \neq y \text{ and } x \nsim y. \end{cases}$$

A Deza graph Δ is called strictly Deza graph, if Δ has diameter 2, and is not SRG.

Distance-regular graphs and Strongly regular graphs

A connected graph Γ is distance-regular if for any vertices x and y of Γ and any integers $i, j = 0, 1, \ldots, d$ (where d is the graph diameter), the number of vertices at distance i from x and distance j from y depends only on i, j, and the graph distance between x and y, independently of the choice of x and y.

$$\bigcup$$

A graph Γ is called strongly regular with parameters (v, k, λ, μ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$:

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k, & \text{if } x = y, \\ \lambda, & \text{if } x \sim y, \\ \mu, & \text{if } x \neq y \text{ and } x \nsim y. \end{cases}$$

Spectra of SRG

The adjacency matrix of a non-trivial (v, k, λ, μ)-strongly regular graph has exactly three eigenvalues:

- eigenvalue k, whose multiplicity is 1; it is called "principal eigenvalue"
- "non-principal" eigenvalues $r > 0$ and $s < 0$, which are roots of the equation

$$x^2 + (\mu - \lambda)x + (\mu - k) = 0$$

Cayley graphs

Let G be a finite group.
Let $S \subset G$ be a nonempty subset with the following properties

- $1_G \notin S$;
- $\forall s \in S \Rightarrow s^{-1} \in S$.

A graph $Cay(G, S)$ whose vertices are the elements of G,
and the adjacency is defined by the following rule

$$x \sim y \Leftrightarrow xy^{-1} \in S, \quad \forall x, y \in G$$

is called a Cayley graph of group G with the connection set S.

Vertex connectivity of SRG, DRG and Cayley graphs

Vertex connectivity of connected k-regular Cayley graph is at least $\frac{2}{3}(k+1)$;

Vertex connectivity of connected strongly regular graph equals its valency. (Brouwer, Mesner'85)

Vertex connectivity of connected distance-regular graph equals its valency. (Brouwer, Koolen'09)

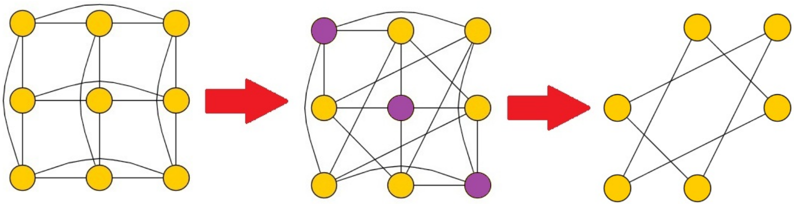
Problem

What is vertex connectivity of strictly Deza graphs?

Construction of Deza graphs

Let Γ be an (n, k, λ, μ)-SRG with $k \neq \mu$ and $\lambda \neq \mu$ and adjacency matrix M. Let P be a permutation matrix. Then PM is the adjacency matrix of a Deza graph Δ if and only if $P = I$ or P represents an involution of Γ that interchanges only nonadjacent vertices. Moreover, Γ is strictly Deza if and only if $P \neq I, \lambda \neq 0$ and $\mu \neq 0$.

Vertex connectivity of Deza graphs obtained from the Construction



Vertex connectivity of the Deza graph obtained from 3×3-lattice is equal to 3.

Menger's Theorem

Theorem (Menger, 1927)

Let Γ be a finite undirected graph. Let x, y be two nonadjacent vertices. Then the minimum number of vertices whose removal disconnects x and y is equal to the maximum number of pairwise vertex-independent paths from x to y.

Vertex connectivity of Deza graphs obtained from Construction: General case

Theorem (Gavrilyuk, Goryainov, Kabanov, 2013)

Let Δ be a Deza graph obtained from strongly regular graph Γ with non-principal eigenvalues r, s. Then one of the following cases holds:

(1) $r > 2$ and $s < -2$, and the vertex connectivity of Δ is equal to its valency;

c (2) $s = -2$, and the vertex connectivity of Δ is equal to its valency, excepting the case when Γ is 3×3-lattice;

(3) $r \leq 2$.

Strongly regular graphs with $s = -2$

Theorem (Seidel)

Let Γ be a strongly regular graph with $s = -2$. Then one of the following cases holds:

(1) Γ *complete multipartite with parts of size* 2*;*

(2) Γ *is* $n \times n$*-lattice with parameters* $(n^2, 2(n-1), n-2, 2)$*;*

(3) Γ *is triangular* $T(n)$ *with parameters* $(n(n-1)/2, 2(n-2),$
$n-2, 4)$*;*

(4) Γ *is Shrikhande graph with parameters of* 4×4*-lattice;*

(5) Γ *is one of three Chang graphs with parameters of* $T(8)$*;*

(6) Γ *is Petersen graph with parameters* $(10, 3, 0, 1)$*;*

(7) Γ *is Clebsch graph with parameters* $(16, 10, 6, 6)$*;*

(8) Γ *is Schlafli graph with parameters* $(27, 16, 10, 8)$*.*

Case $r = 1$ (complement to $s = -2$)

A strongly regular graph Γ has eigenvalue $r = 1$ if and only if the complement to Γ has $\overline{s} = -2$.

So, in the case $r = 1$ we can use Seidel's classification (just take complements).

To apply Contruction to the complements it is sufficient to know automorphisms of Seidel graphs that interchanges only adjacent vertices.

Case $r = 1$ (complement to $s = -2$)

Theorem (Goryainov, Shalaginov, 2013)

For the graph $T(n)$, there exists a unique, up to the indexing of the vertices, order 2 automorphism that interchanges only adjacent vertices.

Theorem (Goryainov, Shalaginov, 2013)

For the $n \times n$-lattice, there exists $\lfloor n/2 \rfloor$, up to the indexing of the vertices, order 2 automorphisms that interchanges only adjacent vertices; each such automorphism permutes rows in t pairs, where $1 \leq t \leq \lfloor n/2 \rfloor$.

Case $r = 1$ (complement to $s = -2$)

Theorem (Goryainov, Panasenko, 2015)

Let Δ be a Deza graph, obtained from strongly regular graph with $r = 1$. Then one of the following cases holds:

(1) Δ *is obtained from one of sporadic graphs (4)-(8); vertex connectivity of* Δ *is equal to its valency;*

(2) Δ *is obtained from complement to triangular graph* $T(n)$, $n \geq 3$*; vertex connectivity of* Δ *is equal to its valency;*

(3) Δ *is obtained from complement to* $n \times n$*-lattice,* $n \geq 3$*; vertex connectivity of* Δ *is equal to* $k - 1$*, where* k *is valency.*

Case $r = 2$

Strongly regular graphs with $r = 2$ were classified by Kabanov, Makhnev and Paduchikh.

For many sets of parameters the question of existence is open.

In paticular, there exists three hypothetically infinite families of parameters.

Thank you for your attention!