On colour-preserving automorphisms of Cayley graphs

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Cayley graph

Let S be a subset of a group G, such that $S = S^{-1}$.
The Cayley graph of G, with respect to S, is the graph $Cay(G; S)$ whose vertices are the elements of G, and with an edge $x \sim xs$, for each $x \in G$ and $s \in S$.

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Edge-colouring of Cayley graphs

$Cay(G; S)$ has a natural edge-colouring. Each edge of the form $x \sim xs$ is coloured with the set $\{s, s^{-1}\}$.

Colour-preserving automorphisms

Colour-preserving automorphism

Let G be a group and S a subset of G such that $S = S^{-1}$. A permutation ϕ of G is a colour-preserving automorphism of $Cay(G; S)$ if and only if we have $\phi(xs) \in \{\phi(x)s^{\pm 1}\}$, for each $x \in G$ and $s \in S$.

Colour-preserving automorphisms - examples

Left translation

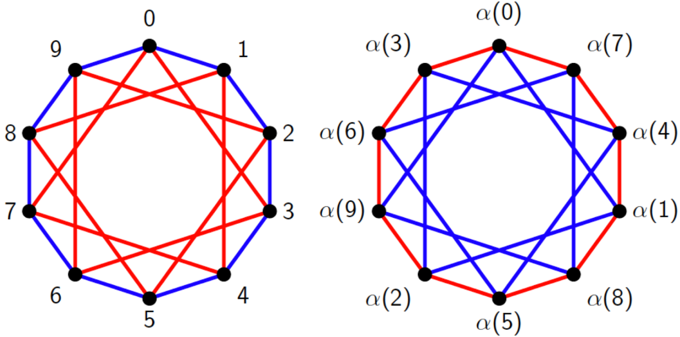
The left translation $x \mapsto gx$ is a colour-preserving automorphism of $Cay(G;S)$.

Automorphisms of G

An automorphism α of G such that $\alpha(s) \in \{s^{\pm 1}\}$ for all $s \in S$ is a colour-preserving automorphism of $Cay(G;S)$.

Observe

that if α is a group automorphism of G, then $\alpha(gs) = \alpha(g)\alpha(s)$, so the colour s maps to the colour $\alpha(s)$. Thus, such an automorphism is at least colour-permuting. Eg.



$\mathrm{Cay}(\mathbb{Z}_{10}; \{1, 9, 3, 7\})$ $\mathrm{Cay}(\mathbb{Z}_{10}; \{\alpha(1), \alpha(9), \alpha(3), \alpha(7)\})$

Affine function & Cayley Colour Automorphism (CCA)

Affine function

A function $\phi\colon G \to G$ is said to affine if it is the composition of an automorphism $\alpha \in Aut(G)$ with left translation by an element of G. That is, $\phi(x) = \alpha(gx)$ for some $\alpha \in Aut(G)$ and $g \in G$.

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CCA Cayley graphs

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CCA groups

A group G is said to be CCA group if every connected Cayley graph on G is CCA.

Q_8 and $\mathbb{Z}_4 \times \mathbb{Z}_2$ are not CCA groups

$$Cay(Q_8; \{\pm i, \pm j\}) \cong Cay(\mathbb{Z}_4 \times \mathbb{Z}_2; \{\pm(1,0), \pm(1,1)\}) \cong K_{4,4}$$

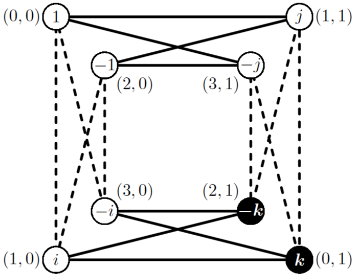
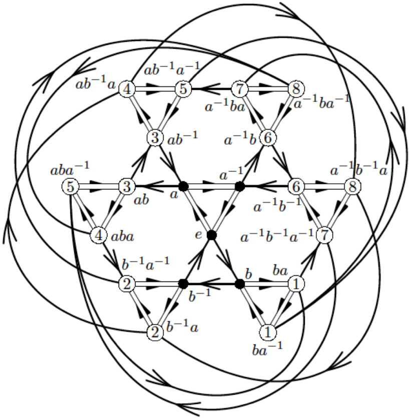


Figure 1: Interchanging the two black vertices while fixing all of the white vertices is a colour-preserving graph automorphism that fixes the identity vertex but is not a group automorphism.

Examples

Nonabelian group of order 21

The nonabelian group of order 21 is not CCA.



Nonabelian group of order 21

The nonabelian group of order 21 is not CCA.

Proof. Let $G = \langle a, x \mid a^3 = e,\ a^{-1}xa = x^2 \rangle$. (Since $x = e^{-1}xe = a^{-3}xa^3 = x^8$, the relations imply $x^7 = e$, so G has order 21.) By letting $b = ax$, we see that G also has the presentation

$$G = \langle a, b \mid a^3 = e,\ (ab^{-1})^2 = b^{-1}a \rangle.$$

As illustrated in Figure every element of G can be written uniquely in the form

$$a^i b^j a^k, \text{ where } i, j, k \in \{0, \pm 1\} \text{ and } j = 0 \Rightarrow k = 0.$$

Define

$$\varphi(a^i b^j a^k) = \begin{cases} b^j a^{-k} & \text{if } i = 0, \\ ab^{-j}a^k & \text{if } i = 1, \\ a^{-1}b^{-j}a^{-k} & \text{if } i = -1. \end{cases}$$

Then φ is a colour-preserving automorphism of $\mathrm{Cay}\big(G; \{a^{\pm 1}, b^{\pm 1}\}\big)$ (see Figure). However, φ is not affine, since it fixes e, but is not an automorphism of G (because $\varphi(ab) = ab^{-1} \neq ab = \varphi(a)\,\varphi(b)$). □

Examples

Wreath product $\mathbb{Z}_m \wr \mathbb{Z}_n$

The wreath product $\mathbb{Z}_m \wr \mathbb{Z}_n$ is not CCA whenever $m \geq 3$ and $n \geq 2$.

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Proof. This group is a semidirect product

$$(\mathbb{Z}_m \times \mathbb{Z}_m \times \cdots \times \mathbb{Z}_m) \rtimes \mathbb{Z}_n.$$

For the generators $a = \big((1, 0, 0, \ldots, 0), 0\big)$ and $b = \big((0, 0, \ldots, 0), 1\big)$, the map

$$\big((x_1, x_2, x_3, \ldots, x_n), y\big) \mapsto \big((-x_1, x_2, x_3, \ldots, x_n), y\big)$$

(negate a single factor of the abelian normal subgroup) is a colour-preserving automorphism of $\mathrm{Cay}\big(\mathbb{Z}_m \wr \mathbb{Z}_n; \{a^{\pm 1}, b^{\pm 1}\}\big)$ that fixes the identity element but is not a group automorphism. $\square$

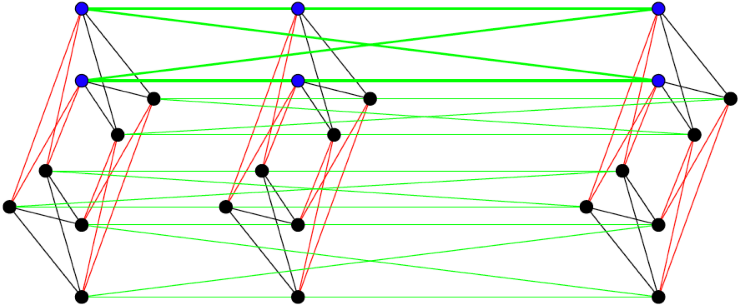
Main results

Theorem 1

- There is a non-CCA group of order n if and only if $n \geq 8$ and n is divisible by either 4, 21, or a number of the form $p^q q$, where p and q are primes.
- An abelian group is not CCA if and only if it has a direct factor that is isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$ or a group of the form $\mathbb{Z}_{2^k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, with $k \geq 2$.
- Every dihedral group is CCA.

$\mathbb{Z}_{2^n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $n \geq 2$ is not CCA

$\mathrm{Cay}(\mathbb{Z}_{2^n} \times \mathbb{Z}_2 \times \mathbb{Z}_2; \{\pm(1,0,0), \pm(2^{n-2},1,0), \pm(2^{n-2},0,1)\})$ *is not CCA when* $n \geq 2$.



Main results

Proposition

- Every non-CCA group G of odd square-free order has a section that is isomorphic to (unique) nonabelian group F_{21} of order 21.
- There exists a unique non-CCA Cayley graph Γ of F_{21}.
- If $Cay(G,S)$ is a non-CCA graph of a group G of odd square-free order, then $G = H \times F_{21}$ for some CCA group H, and $Cay(G;S) = Cay(H;T)\square\Gamma$.

$Cay(F_{21}; \{a^{\pm 1}, (ax)^{\pm 1}\})$ the unique non-CCA graph of F_{21}

$$F_{21} = \langle a, x \mid a^3 = x^7 = e, a^{-1}xa = x^2 \rangle$$

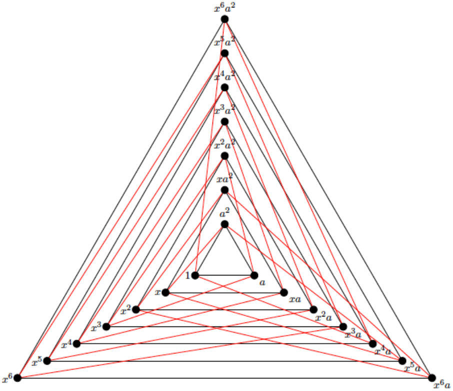


Figure 1. The unique non-CCA Cayley graph of F_{21}.

Main results

Let A be an abelian group of even order. Choose an involution y of A. The corresponding generalized dicyclic group is

$$Dic(y, A) = \langle x, A \mid x^2 = y, x^{-1}ax = a^{-1}, \text{ for all } a \in A\rangle.$$

For $n \geq 1$, a semidihedral (or quasidihedral) group is a group

$$SemiD_{16n} = \langle a, x \mid a^{8n} = x^2 = e, xa = a^{4n-1}x\rangle.$$

Theorem 2

- No generalized dicyclic group or semidihedral group is CCA, except that $\mathbb{Z}_4$ is dicyclic, but is CCA.

Main results

Let α be an automorphism of a group A, and let $n \in \mathbb{Z}^+$. Then we can define an automorphism $\tilde{\alpha}$ of A^n by

$$\tilde{\alpha}(w_1, \ldots, w_n) = (\alpha(w_n), w_1, \ldots, w_{n-1}).$$

It is easy to see that the order of $\tilde{\alpha}$ is n times the order of α, and so we may form the corresponding semidirect product $A^n \rtimes \mathbb{Z}_{n|\alpha|}$, called the semi-wreathed product of A by $\mathbb{Z}_n$ with respect to α.

Theorem 3

- Every non-CCA group of odd order has a section that is isomorphic to either the nonabelian group of order 21 or a certain generalization of a wreath product (called semi-wreathed product).

Main results

Theorem 4

- If $G \times H$ is CCA then G and H are both CCA. The converse is not always true (for example, $\mathbb{Z}_4 \times \mathbb{Z}_2$ is not CCA), but it does hold if $gcd(|G|, |H|) = 1$.

Colour-permuting automorphisms

An automorphism α of a Cayley graph $Cay(G; S)$ is colour-permuting if it respects the colour classes; that is, if two edges have the same colour, then thier images under α must also have the same colour. That is, there exists a permutation π of S s.t.

$$\alpha(gs) \in \{\alpha(g)\pi(s)^{\pm 1}\}$$

for all $g \in G$ and $s \in S$.

Strongly CCA groups

A group G is strongly CCA if every colour-permuting automorphism of every connected Cayley graph on G is affine.

Strongly CCA groups

Any strongly CCA group is CCA.
Converse is not true.
For example, any dihedral group is CCA but it is not strongly CCA if its order is of the form $8k + 4$. However, the converse does hold for abelian groups and groups of odd order.

Proposition

A CCA group is strongly CCA if either it is abelian or it has odd order.

Questions

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Are there other natural colourings for which we could ask this question?

Thank you!