ON GRUENBERG–KEGEL GRAPHS OF FINITE GROUPS

Anatoly S. Kondrat'ev and Natalia V. Maslova

based on joint works with O. Alekseeva, I. Gorshkov, and D. Pagon

Novosibirsk: G2S2, August 17, 2016

Plan of this talk

- Some background
- Anatoly Kondrat'ev: Finite groups whose Gruenberg–Kegel graphs are triangle–free (joint work with O. Alekseeva)
- Natalia Maslova: On realizability of a graph as Gruenberg–Kegel graph of a group (based on joint works with I. Gorshkov and D. Pagon)

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We use the term ”group” while meaning ”finite group”.

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Definitions and Notation

A clique (resp. coclique) is an undirected graph such that every two distinct vertices are adjacent (resp. non-adjacent).

A clique (resp. coclique, resp. chain) with n vertices is called n-clique (resp. n-coclique, resp. n-chain).

A triangle is a 3-cycle.

A tree is a graph which is connected and has no cycles.

Let Γ and Δ be graphs.

Γ is Δ-free if Γ does not contain an induced subgraph isomorphic to Δ.

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Recall, a non-trivial group G is simple if it doesn't contain nontrivial proper normal subgroups.

Simple groups were classified in 1980. With respect to this classification, non-abelian simple groups are:

- Alternating groups A_n for $n \geq 5$;
- Classical groups of Lie type: $PSL_n(q)$, $PSU_n(q)$, $PSp_{2n}(q)$, $P\Omega_n(q)$ (n is odd), $P\Omega_n^+(q)$ (n is even), $P\Omega_n^-(q)$ (n is even);
- Exceptional groups of Lie type: $E_8(q)$, $E_7(q)$, $E_6(q)$, ${}^2E_6(q)$, ${}^3D_4(q)$, $F_4(q)$, ${}^2F_4(q)$, $G_2(q)$, ${}^2G_2(3^{2k+1})$, ${}^2B_2(2^{2k+1})$;
- 26 sporadic groups.

A group G is almost simple, if $S \cong Inn(S) \trianglelefteq G \leq Aut(S)$, where S is a non-abelian simple group.

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Definitions and Notation

A cyclic group of order n is denoted by C_n.

A group G is a Frobenius group if there is a non-trivial subgroup C of G such that $C \cap gCg^{-1} = \{1\}$ whenever $g \notin C$. C is a Frobenius complement of G.

$K = \{1\} \cup (G \setminus \bigcup_{g \in G} gCg^{-1})$ is the Frobenius core of G. K is a normal subgroup of G.

$S(G)$ is the solvable radical of G, i. e. the largest normal solvable subgroup of G.

$F(G)$ is the Fitting subgroup of G, i. e. the largest normal nilpotent subgroup of G.

$O_p(G)$ is the largest normal p-subgroup of G.

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Definitions

Let G be a group.

The spectrum $\omega(G)$ is the set of all element orders of G.

The prime spectrum $\pi(G)$ is the set of all prime elements of $\omega(G)$ (i. e., the set of all prime divisors of $|G|$).

A graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$ is called the Gruenberg–Kegel graph or the prime graph of G.

If p and q are primes then $pq \in \omega(G)$ if and only if there exist $x, y \in G$ such that $|x| = p$, $|y| = q$ and $xy = yx$.

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Examples

Let $\pi(G) = \{p_1, p_2, p_3, p_4, p_5\}$

and $G = C_n$, where $n = p_1p_2p_3p_4p_5$

G is abelian $\Rightarrow$

if $p_i, p_j \in \omega(G)$ and $p_i \neq p_j$ then $p_ip_j \in \omega(G)$

$\Gamma(G)$:

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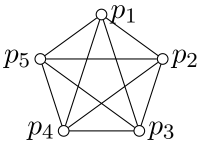

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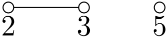

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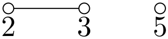

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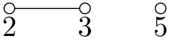

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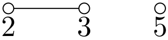

$$\omega(A_5) = \{1, 2, 3, 5\}$$
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$\Gamma(A_5) = \Gamma(A_6)$:

G. Higman (1957) showed that if G is a solvable non-primary group whose element orders are prime powers (so called *EPPO*-group) then $|\pi(G)| = 2$ and G is either a Frobenius group or a 2-Frobenius group (i. e., $G = ABC$, where A and AB are normal subgroups of G, AB and BC are Frobenius groups with cores A and B and complements B and C, respectively).

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Suzuki (1962) proved that a non-abelian simple $EPPO$-group is isomorhic to $L_2(q)$ for $q \in \{5, 7, 8, 9, 17\}$, $L_3(4)$, $Sz(8)$ or $Sz(32)$.

G. Higman (1968) proved that if G is a non-solvable non-simple $EPPO$-group then either $G \cong M_{10}$, or $G/O_2(G)$ is isomorphic to $L_2(q)$ for $q \in \{4, 8\}$ or $Sz(q)$ for $q \in \{8, 32\}$, where $O_2(G)$ is known.

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If G is $EPPO$-group then $\Gamma(G)$ is a coclique;

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Question. What are finite groups whose Gruenberg–Kegel graphs are triangle–free?

Lemma. Let G be a non-solvable group whose Gruenberg–Kegel graph is triangle–free. Then $G/S(G)$ is an almost simple group, where $S(G)$ is a solvable radical of G (i.e. the largest normal solvable subgroup of G).

Theorem 1 (O. Alekseeva, A. Kondrat'ev, 2015). Let G be an almost simple group whose Gruenberg–Kegel graph is triangle–free. Then G is known.

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Realizability of a graph as Gruenberg–Kegel graph

Question. Let Γ be a graph whose vertices are numbers form finite set π of primes. Is there a group G such that $\Gamma = \Gamma(G)$?

Negative solution in general case.

Let Γ be a 3-coclique with $V(\Gamma) = \{p, q, r\}$, where p, q and r are pairwise distinct odd primes.

Assume, G is a group such that $\Gamma(G) = \Gamma$.

Then by Feit-Thompson theorem G is solvable.

A contradiction with Lucido's result.

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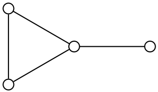

$\Gamma(Aut(J_2))$:

$\Gamma(A_{10})$:

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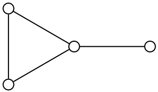

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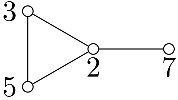

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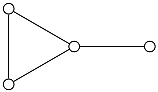

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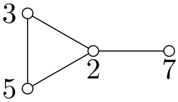

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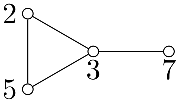

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Negative solution in general case gives an n-coclique, where $n \geq 5$.

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Question (A. L. Gavrilyuk, 2012). Let Γ be a small graph. Is Γ realizable as Gruenberg–Kegel graph of a group?

Theorem (A. L. Gavrilyuk, I. V. Khramtsov, A. S. Kondrat'ev and N. M., 2014). Let Γ be a graph with at most 5 vertices. Then

(1) if Γ is 5-coclique then Γ is not realizable as Gruenberg–Kegel graph of a group;

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Let Γ contains a 3-coclique.

If Γ is realizable as Gruenberg–Kegel graph of a group G then G is non-solvable.

If Γ is a complete bipartite graph $K_{m,n}$ then Γ is triangle–free.

Lemma. Let G be a non-solvable group whose Gruenberg–Kegel graph is triangle–free and $S(G)$ be the largest normal solvable subgroup of G. Then $G/S(G)$ is an almost simple group.

Lemma (N. M. and D. Pagon, 2015). Let Γ be a complete bipartite graph $K_{m,n}$. Then Γ is realizable as Gruenberg–Kegel graph of a group if and only if $m + n \leq 6$ and $(m, n) \neq (3, 3)$.

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Question. Let Γ is realizable as Gruenberg–Kegel graph of a group G. How many "ways" are there to choice G?

$\Gamma(S_5)$:

$\Gamma(S_5 \times C_p)$, where $p \notin \{2,3,5\}$ is a prime:

$\pi(S_5 \times C_{p_1}) \neq \pi(S_5 \times C_{p_2})$, where p_1 and p_2 are distinct primes.

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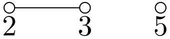

$\Gamma(S_5 \times C_p)$, where $p \notin \{2, 3, 5\}$ is a prime:

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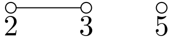

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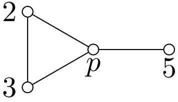

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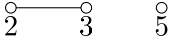

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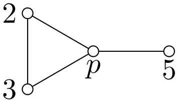

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$\Gamma(Q \times Aut(J_2))$, where Q is a 2-group:

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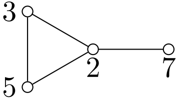

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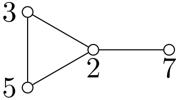

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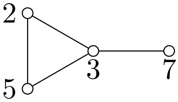

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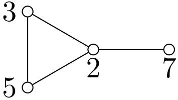

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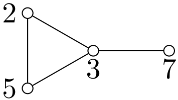

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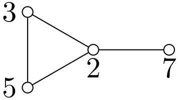

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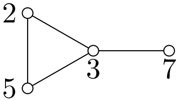

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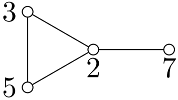

Theorem (A. Kondrat'ev, 2012). Let G be a group and $\Gamma(G) = \Gamma(Aut(J_2))$. Then G is known.

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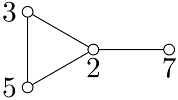

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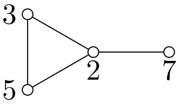

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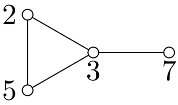

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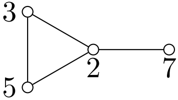

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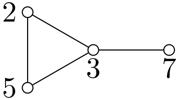

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Theorem 4 (N. M. and D. Pagon, 2016). Let Γ be a complete bipartite graph $K_{m,n}$, where $m \leq n$. Then the following statements hold:

(1) Γ is realizable as Gruenberg–Kegel graph of a group if and only if $m + n \leq 6$ and $(m, n) \neq (3, 3)$;

(2) if $m + n \leq 6$ and $(m, n) \notin \{(3, 3), (1, 5), (5, 1)\}$ then there exist infinitely many sets T of primes such that Γ is realizable as Gruenberg–Kegel graph of a group G and $T = \pi(G)$;

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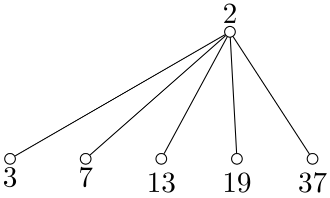

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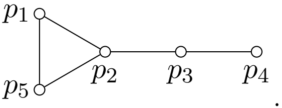

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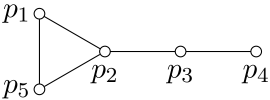

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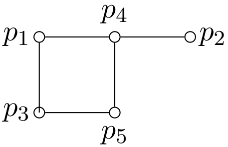

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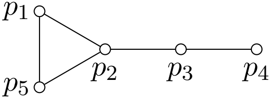

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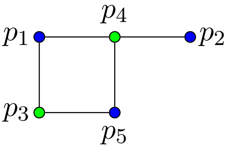

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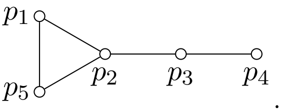

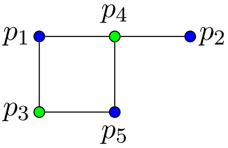

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Realizability of a graph as Gruenberg–Kegel graph

Theorem 5 (I. B. Gorshkov and N. M., 2016). Almost simple groups whose Gruenberg–Kegel graphs coincide with Gruenberg–Kegel graphs of solvable groups are known.

Corollary. Let G be an almost simple group. Then the following conditions are equivalent:

(1) $\Gamma(G)$ doesn't contain a 3-coclique;

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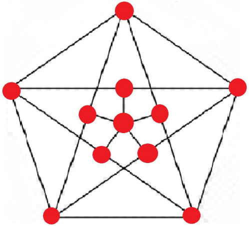

Figure 1: Grotzsch graph

Theorem 6 (I. B. Gorshkov and N. M., 2016). The complement to the Grotzsch graph is not realizable as Gruenberg–Kegel graph of a group.

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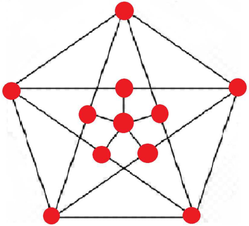

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Thank you!