On a Characterization of the Grassmann Graphs $J_q(2d+2, d)$

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based on joint work with **Jack Koolen**
USTC (Hefei, China)

G2S2, Novosibirsk, 2016

The Grassmann graph $J_q(n, d)$

- Let $q \geq 2$ be a prime power, $n \geq d \geq 1$ be integers.
- $J_q(n, d)$ has as vertices all d-dim. subspaces $U \subseteq \mathbb{F}_q^n$.
- $U_1 \sim U_2$ iff $dim(U_1 \cap U_2) = d - 1$.
- $J_q(n, d) \cong J_q(n, n - d)$, diameter equals $\min(d, n - d)$.

  $\Rightarrow$ w.l.o.g., we assume $n \geq 2d$.
- Distance-transitive $\Rightarrow$ Distance-regular graph (**DRG**).
- Q-**polynomial**.

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Classification problem of Q-DRG

Bannai's problem (early 1980's)

Can we classify the Q-polynomial distance-regular graphs with large diameter?

(Bannai, Ito, *Algebraic combinatorics I: Association schemes.*)

$$\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}.$$

Q-polinomiality can be recognized from $\iota(\Gamma)$.

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Distance-regular graphs

For a distance-regular graph Γ with diameter D, its **intersection array** is:

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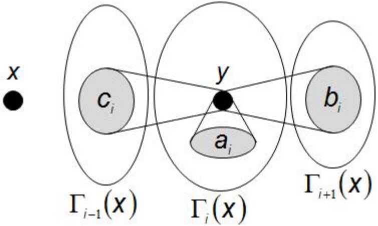
where, for $\forall$ pair of vertices x, y with $d(x,y) = i$,

$$|\Gamma_1(y) \cap \Gamma_{i-1}(x)| = c_i, \;\; |\Gamma_1(y) \cap \Gamma_i(x)| = a_i, \;\; |\Gamma_1(y) \cap \Gamma_{i+1}(x)| = b_i$$



Γ is **regular** with valency $b_0 = |\Gamma_1(x)|$, $\forall x \in V(\Gamma)$ so that

$$b_0 = c_i + a_i + b_i.$$

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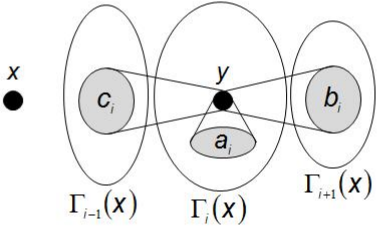

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Characterization of $J_q(n,d)$ by $\iota(J_q(n,d))$

Theorem (Metsch, 1995)

The Grassmann graph $J_q(n,d)$, $d > 2$, is characterized by its intersection array with the following *possible exceptions*:

- $n = 2d$ or $n = 2d + 1$,
- $n = 2d + 2$ if $q \in \{2, 3\}$,
- $n = 2d + 3$ if $q = 2$.

Some recent progress:

- Van Dam, Koolen (2004): an actual exception for $J_q(2d+1, d)$, the *twisted Grassmann graph*.
- G., Koolen (2014+): no exceptions for $J_2(2d, d)$, if d is odd or large enough.

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Local structure of the Grassmann graphs

$\Gamma_1(x)$ = the **local graph** of a vertex x of a graph Γ.

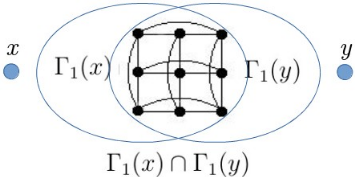
$\Gamma = J_q(n,d)$, $U \in \Gamma$: $\quad U \subset (d+1)$ and $(d-1) \subset U$

$\Gamma_1(U)$ = q-clique extension of $\begin{bmatrix} n-d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}$-lattice

where $\begin{bmatrix} n \\ 1 \end{bmatrix} := (q^n - 1)/(q-1)$.

μ-**graph** of x, y (at distance 2) = $\Gamma_1(x) \cap \Gamma_1(y)$.

Every μ-graph in the Grassmann graphs $J_q(n,d)$ is the $(q+1) \times (q+1)$-lattice.



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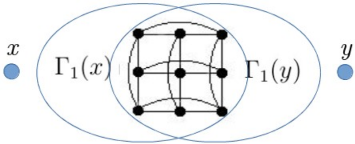

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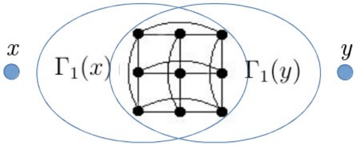

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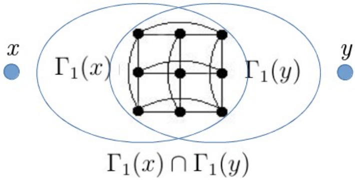

Local characterization of $J_q(n, d)$

Theorem (Numata, Cohen, [BCN, Thm 9.3.8])

Let Γ be a finite connected graph such that

- if $x, y \in \Gamma$ are at distance 2 then
$$\Gamma_1(x) \cap \Gamma_1(y) \text{ is a lattice,}$$
- if $x, y, z \in \Gamma$ form a coclique then
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Then Γ is either a clique, or a Johnson graph, or the folded Johnson graph $J(2d, d)$, or a Grassmann graph over a finite field.

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Theorem (G., Koolen, 2014+)

The Grassmann graph $J_2(2d, d)$, $d > 2$, is characterized by its intersection array, if at least one of the following holds:

- the diameter d is odd,
- the diameter d is large enough.

Proof:

Suppose that $\iota(\Gamma) = \iota(J_2(2d, d))$.

(1) The Terwilliger algebra theory to derive some local properties of Γ.

(2) The Hoffman graphs theory to determine the local graphs of Γ.

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Suppose that $\iota(\Gamma) = \iota(J_q(2d, d))$.

- Use the Terwilliger algebra theory to show that the local graphs of Γ are *cospectral* to the local graphs of $J_q(2d, d)$ *(for any q)*.

  Suppose further that $q = 2$.

- Use the Terwilliger algebra theory to show that the μ-graphs of Γ are the same as of $J_2(2d, d)$, if d is odd.
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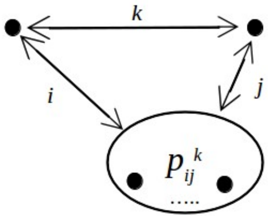
Intersection numbers

Let Γ be a DRG.
Let x, y be any pair of vertices of Γ with $d(x, y) = k$.
Then the **intersection numbers** of Γ

$$p^k_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)| = |\Gamma_j(x) \cap \Gamma_i(y)|$$

do not depend on the choice of x, y, $\forall\ 0 \le i, j, k \le D(\Gamma)$.



Note that $c_i = p^i_{1,i-1}$, $a_i = p^i_{1,i}$, $b_i = p^i_{1,i+1}$.

Triple intersection numbers

Let Γ be a Q-polynomial DRG with diameter $D \geq 3$.
Fix a triple of vertices x, y, z such that $x \sim y$, $x \sim z$.
Denote a **triple intersection number**

$$[\ell, m, n] := [\ell, m, n]_{x,y,z} = |\Gamma_\ell(x) \cap \Gamma_m(y) \cap \Gamma_n(z)|$$

Terwilliger (1995) and Dickie (1996) showed that for $i \geq 2$

$$[i, i-1, i-1] = \kappa_{i,\delta}[1, 1, 1] + \tau_i$$

where $\delta = d(y, z) \in \{1, 2\}$, and $\kappa_{i,\delta}$ and τ_i are real scalars that do not depend on x, y, z.

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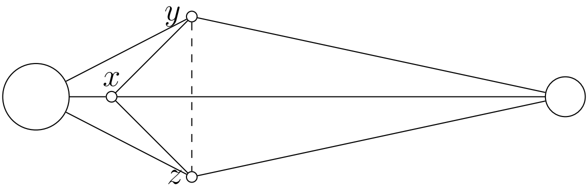

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In the same manner, one can show that

$$[i, i+1, i+1] = \sigma_{i,\delta}[1,1,1] + \rho_i$$

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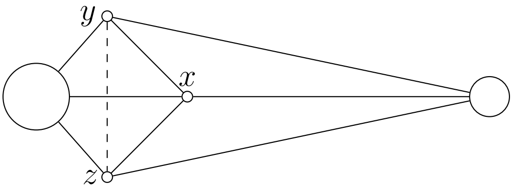

Bose-Mesner algebra

- Let Γ be a DRG with diameter D and on v vertices.
- Define the **distance-i matrix** A_i of Γ:

$$(A_i)_{x,y} := \begin{cases} 1 \text{ if } d(x,y) = i, \\ 0 \text{ if } d(x,y) \neq i. \end{cases}$$

- A_1 — the adjacency matrix of Γ, $A_0 = I$.
- One has:

$$A_i A_j = A_j A_i = \sum_{k=0}^{D} p_{ij}^k A_k,$$

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1},$$

where p_{ij}^k are the intersection numbers.

- The **Bose-Mesner** algebra $\mathcal{M} \subseteq \mathbb{C}^{v \times v}$ is the matrix algebra over $\mathbb{C}$ generated by $A_0 = I, A_1, \ldots, A_D$.

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Terwilliger algebra

Fix any vertex $x \in V(\Gamma)$.

For $0 \leq i \leq D$, denote by $E_i^* := E_i^*(x)$ a diagonal matrix with rows and columns indexed by $V(\Gamma)$, and defined by

$$(E_i^*)_{y,y} := \begin{cases} 1 \text{ if } d(x,y) = i, \\ 0 \text{ if } d(x,y) \neq i. \end{cases}$$

The **dual Bose-Mesner** algebra (w.r.t. x)

$$\mathcal{M}^* := \mathcal{M}^*(x) = span\{E_0^*, E_1^*, \ldots, E_D^*\}.$$

The **Terwilliger** (or **subconstituent**) algebra (w.r.t. x)

$$\mathcal{T} := \mathcal{T}(x) = \langle \mathcal{M}, \mathcal{M}^* \rangle,$$

where $\mathcal{M}$ is the Bose-Mesner algebra of Γ.

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Denote $\widetilde{A} := E_1^* A_1 E_1^*$ and $\widetilde{J} := E_1^* J E_1^*$. One can see

$$\widetilde{A} = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}.$$

where N — the adjacency matrix of $\Gamma_1(x)$.

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The Terwilliger polynomial of a Q-DRG

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- Terwilliger (early 1990's): There exists a polynomial p_T of degree 4 such that, for any vertex $x \in \Gamma$, and any non-principal eigenvalue η of $\Gamma_1(x)$ we have $p_T(\eta) \geq 0$.
- p_T only depends on the intersection numbers of Γ and the Q-polynomial ordering of primitive idempotents of its Bose-Mesner algebra.
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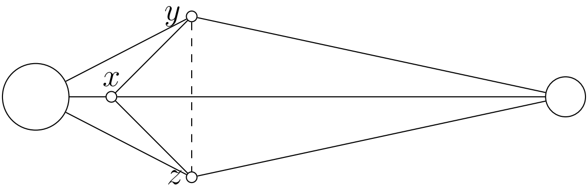
Terwilliger algebra theory: Summary, 1

For a Q-DRG Γ and a base vertex $x \in \Gamma$:

- Triple intersection numbers:

$$[i, i-1, i-1] = \kappa_{i,\delta}[1,1,1] + \tau_i$$



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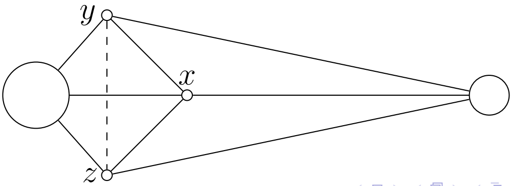
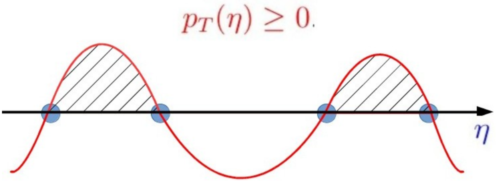


Terwilliger algebra theory: Summary, 2

- The Terwilliger polynomial p_T (of degree 4) such that $p_T(\eta) \geq 0$ for any non-principal eigenvalue η of $\Gamma_1(x)$.



This restricts possible eigenvalues of $\Gamma_1(x)$.

Recall: Sketch of the proof

Suppose that $\iota(\Gamma) = \iota(J_q(2d, d))$.

- Use the Terwilliger algebra theory to show that the local graphs of Γ are *cospectral* to the local graphs of $J_q(2d, d)$ (for any q).

  We use the Terwilliger polynomial.

Suppose further that $q = 2$.

- Use the Terwilliger algebra theory to show that the μ-graphs of Γ are the same as of $J_2(2d, d)$, if d is odd.

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Local graphs of Γ

Theorem (G., Koolen, 2014)

Let Γ be a DRG with the same intersection array as $J_q(2d, d)$, $d \geq 3$. Then, for every vertex $x \in \Gamma$, its local graph $\Gamma_1(x)$ has the same spectrum as the q-clique extension of the $\begin{bmatrix} d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}$-lattice.

Proof: the Terwilliger polynomial + some counting.

This result gives a very strong evidence that $J_q(2d, d)$ is unique, and leads to the following

Problem

Spectral characterization of the clique extensions of lattices.

Negative example: the 3-clique extension of 3×3-lattice has a cospectral mate (Van Dam).

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Using the Hoffman graphs theory, Koolen and co-authors (Yang, Kim (2015); Yang, Yang (2016); Abiad,Yang (201?)) developed a structure theory for graphs with smallest eigenvalue -3.

In particular, their results yield that the 2-clique extension of the $t \times t$-lattice with $t \ggg 0$ is characterized by its spectrum. Together with the Numata-Cohen theorem, we obtain:

Theorem (G., Koolen, 2014+)

The Grassmann graph $J_2(2d, d)$, $d > 2$, is characterized by its intersection array, if at least one of the following holds:

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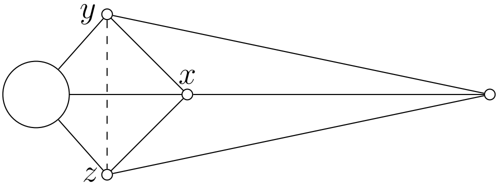
Back to triple intersection numbers

For a Q-DRG Γ, we have that:

$$[i, i+1, i+1] = \sigma_{i,\delta}[1,1,1] + \rho_i$$

where $\delta = d(y,z) \in \{1,2\}$, and $\sigma_{i,\delta}$ and ρ_i are real scalars that do not depend on x, y, z.



If $\iota(\Gamma) = \iota(J_q(2d,d))$ or $\iota(\Gamma) = \iota(J_q(2d+2,d))$ and the diameter d is *odd*, then $\sigma_{i,\delta}$ turns to be *non-integer*.

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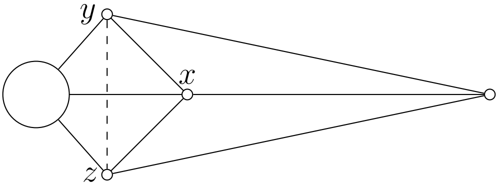

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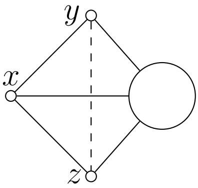

$$y \not\sim z:$$
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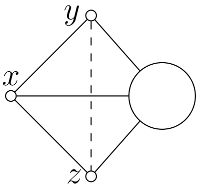

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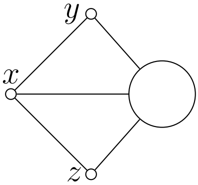

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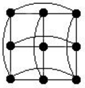
In other words, the μ-graph, say Σ, of y and z is a graph on $c_2 = 9$ vertices, whose valencies belong to $\{1, 4, 7\}$.

$\Sigma \cong$  ?

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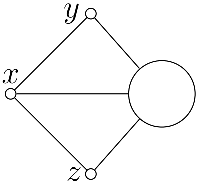

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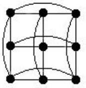
$\Sigma \cong$  ?

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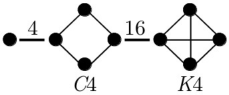
We distinguish between two cases:

- if Σ does not contain 3 pairwise non-adjacent vertices (a 3-coclique), one can easily find this graph:



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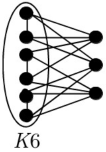

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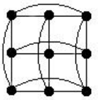

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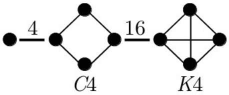

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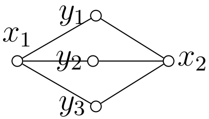

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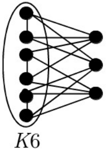

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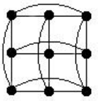

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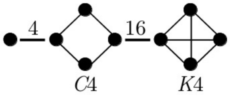
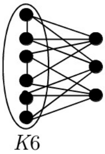
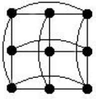
We have the 3 possible μ-graphs:



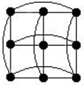




One can easily get rid of the first graph.

To exclude the second graph, we use the fact that $\Gamma_1(x)$ is *cospectral* to the 2-clique extension of the $\begin{bmatrix} d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}$-lattice.

This graph has only 4 distinct eigenvalues $\Rightarrow$ we may compute the number of triangles and quadrangles through any vertex of $\Gamma_1(x)$. Then some counting leaves us with the only possibility:



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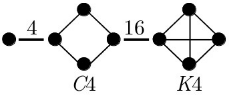
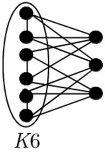
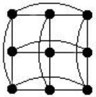
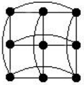






One can easily get rid of the first graph.
To exclude the second graph, we use the fact that $\Gamma_1(x)$ is *cospectral* to the 2-clique extension of the $\begin{bmatrix} d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}$-lattice.

This graph has only 4 distinct eigenvalues $\Rightarrow$ we may compute the number of triangles and quadrangles through any vertex of $\Gamma_1(x)$. Then some counting leaves us with the only possibility:



$\iota(\Gamma) = \iota(J_q(2d, d))$, $q = 2$, odd d

Theorem (G., Koolen, 2014+)

The Grassmann graph $J_2(2d, d)$, $d > 2$, is characterized by its intersection array, if at least one of the following holds:

- the diameter d is odd,
- the diameter d is large enough.

Overview of this talk

- Local structure of $J_q(n, d)$.
- Local characterization of $J_q(n, d)$ by Numata-Cohen.
- Sketch of our characterization of $J_2(2d, d)$.
- The Terwilliger algebra theory.
- What is a problem with $J_q(2d + 1, d)$?
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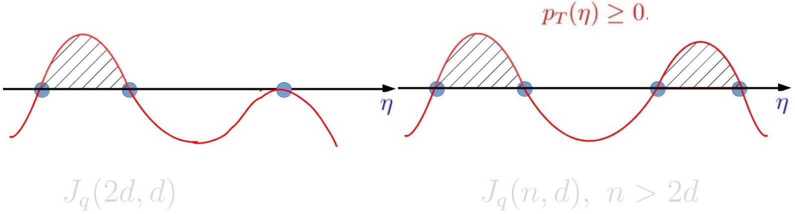
- No enough information from the Terwilliger polynomial.



- No restrictions on triple intersection numbers:

$$[i,i+1,i+1] = \sigma_{i,\delta}[1,1,1] + \rho_i$$

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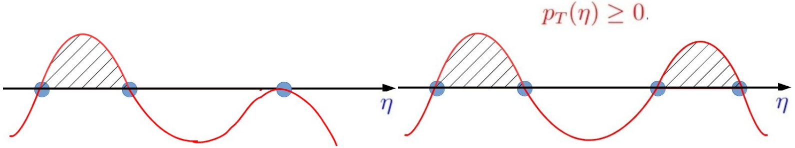

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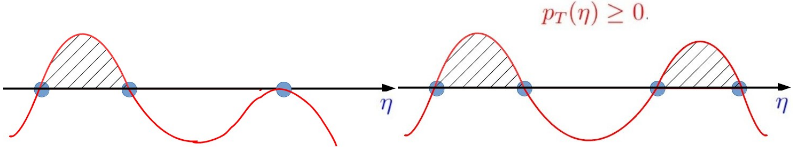

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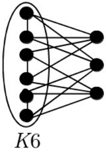
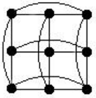
Assuming that d is *odd* and $\iota(\Gamma) = \iota(J_2(2d+2, d))$, we have only 2 possible μ-graphs in Γ:





However, this time, we do not know the spectrum of $\Gamma_1(x)$. But we know that its smallest eigenvalue is at least -3.

So, we can use the Hoffman graphs theory, and this will cost us one more condition: $d \ggg 0$.

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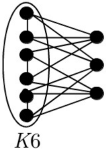

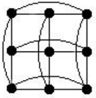

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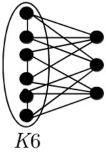

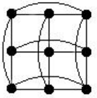

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Hoffman graphs: definitions

- A **Hoffman** graph $\mathfrak{h}$ is a pair (H, ω) of a graph $H = (V, E)$ and a labelling map $\omega : V \to \{f, s\}$, satisfying the following conditions:
  - (i) every vertex with label f is adjacent to at least one vertex with label s;
  - (ii) vertices with label f are pairwise non-adjacent.
- A vertex with label s is called a **slim** vertex; A vertex with label f is called a **fat** vertex; $V_s = V_s(\mathfrak{h})$ – the set of slim vertices of $\mathfrak{h}$; $V_f = V_f(\mathfrak{h})$ – the set of fat vertices of $\mathfrak{h}$.
- If every slim vertex has at least t fat neighbors, we call $\mathfrak{h}$ t-**fat**.
- The **slim graph** of a Hoffman graph $\mathfrak{h}$ is the subgraph of H induced on $V_s(\mathfrak{h})$.

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Representation of Hoffman graphs

For a Hoffman graph $\mathfrak{h}$ and a positive integer n, a mapping $\phi : V(\mathfrak{h}) \to \mathbb{R}^n$ such that:

$$(\phi(x), \phi(y)) = \begin{cases} m & \text{if } x = y \in V_s(\mathfrak{h}), \\ 1 & \text{if } x = y \in V_f(\mathfrak{h}), \\ 1 & \text{if } x \sim y, \\ 0 & \text{otherwise,} \end{cases}$$

is called a **representation of norm** m.

Lemma (Jang, Koolen, Munemasa, Taniguchi)

A Hoffman graph with the smallest eigenvalue at least $-m$ has a representation of norm m. Moreover, w.l.o.g., ϕ can be chosen in such a way that the images of the fat vertices under ϕ are the **unit** vectors (i.e., $(1,0)$-vectors of norm 1).

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KYY theorem

Theorem (Koolen, Yang, Yang, 2016)

There exists a positive integer K such that if a graph Δ has the smallest eigenvalue at least -3 and for $\forall$ vertex $x \in \Delta$:

- (its valency) $k(x) > K$;
- A 5-plex containing x has order at most $k(x) - K$,

then Δ is the slim graph of a 2-fat $\{$, , $\}$-line Hoffman graph.

This simply means that Δ is the slim graph of a Hoffman graph $\mathfrak{d}$, which is an induced Hoffman subgraph of the direct sum $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \ldots$, where $\mathfrak{h}_i$ is isomorphic to an induced Hoffman subgraph of some Hoffman graph from the set $\{$, , $\}$, where $\mathfrak{d}$ and $\mathfrak{h}$ have the same slim graph.

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Suppose that $\iota(\Gamma) = \iota(J_2(2d+2,d))$.
Pick a vertex $x \in \Gamma$ and consider its local graph $\Gamma_1(x)$.

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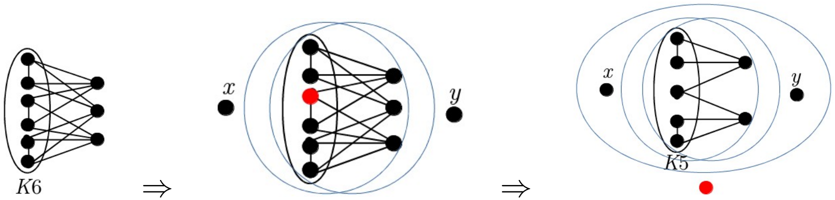
Representation of $\Gamma_1(x)$: contradiction

Now we apply this observation to the wrong μ-graph:



The subgraph induced on x, y and their μ-graph in the local graph of the **red** vertex has an integral representation of norm 3, which is unique.

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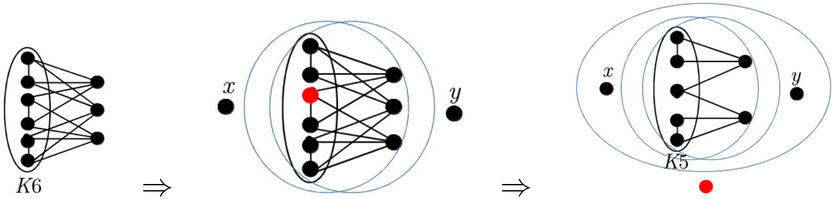

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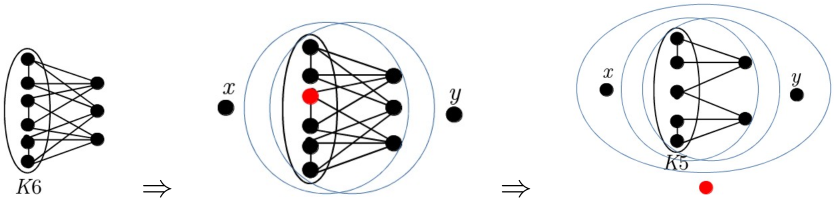

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Summary

Theorem (Metsch, 1995)

The Grassmann graph $J_q(n, d)$, $d > 2$, is characterized by its intersection array with the following *possible exceptions*:

- $n = 2d$ or $n = 2d + 1$,
- $n = 2d + 2$ if $q \in \{2, 3\}$,
- $n = 2d + 3$ if $q = 2$.

Theorem (G., Koolen, 2014)

The Grassmann graph $J_2(2d, d)$, $d > 2$, is characterized by its intersection array, if the diameter d is odd **or** large enough.

Theorem (G., Koolen, 2016)

The Grassmann graph $J_2(2d + 2, d)$, $d > 2$, is characterized by its intersection array, if the diameter d is odd **and** large enough.

Thank you!

$C\pi acu\delta o$!

Xiè-Xiè!