Graphs with Integral Spectrum

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Title: Graphs with Integral Spectrum

Abstract: The spectrum of a graph is the set of eigenvalues of the adjacency matrix of the graph, together with their multiplicities. In 1974, Harary and Schwenk initiate the study of graphs with integral spectra, that is, graphs whose eigenvalues are all integral.
In this talk, we will look at integral Cayley graphs and highlight some open problems [1]. One question is whether the Cayley graph obtained from the symmetric group with respect to the generators of the form $(1, i)$ $i = 2, \ldots, n$ is integral. This graph is known as the star graph. A connection to the representation theory of the symmetric group is explored [2].

A. Abdollahi, E. Vatandoost, Which Cayley graphs are integral? *Electron. J. Combin.* **16(1)** (2009) 17.

Laszlo Babai, Spectra of Cayley Graphs. *Journal of Combinatorial Theory* **27 B** (1979) 180–189.

In Spring 2015, I taught a graduate class on Combinatorics at my university. I taught a few things about algebraic graph theory. At that time, I came across a problem in Cayley graphs that sounded interesting.

Definition (Cayley graph)

Let G be a group, written multiplicatively. Let $S \subseteq G \setminus \{1\}$, with $S^{-1} = \{s^{-1} \mid s \in S\} = S$.
The *Cayley graph* $\mathrm{Cay}(G, S)$ is the graph whose vertices are the elements of G. Two vertices g and h are connected if $hg^{-1} \in S$, i.e., if $h = sg$ for some $s \in S$. So, edges are of the form (g, sg).

In this talk, the groups act on the right.

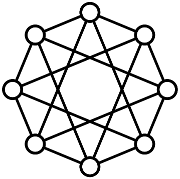
We do not require $\langle S \rangle = G$. So, our Cayley graphs could be disconnected (and there is one instance where they are).

The *spectrum of a graph* is the set of eigenvalues together with their multiplicities.

1. The eigenvalues are real numbers because the djacency matrix is a symmetric real matrix.
2. The eigenvalues do not depend on the ordering of vertices.

Example

$\mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3\}) \simeq K_{4,4}$ is



The adjacency matrix is

$$\begin{bmatrix} 0&1&0&1&0&1&0&1 \\ 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \\ 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \\ 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \\ 1&0&1&0&1&0&1&0 \end{bmatrix}$$

Spectrum:

$$\sigma = \{4^1,\ 0^6,\ (-4)^1\}.$$

The eigenvalues are all integers!

Consider the following question (Harary and Schwenk, 1974):

Which graphs are integral ?

Frank Harary (1921-2005)

The question asks which graphs have a spectrum consisting solely of integers.

Another graph:

Example

$\mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$

Adjacency matrix

$$\begin{bmatrix} 0&1&1&0&0&0&1&1\\ 1&0&1&1&0&0&0&1\\ 1&1&0&1&1&0&0&0\\ 0&1&1&0&1&1&0&0\\ 0&0&1&1&0&1&1&0\\ 0&0&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&1&0&0&0&1&1&0 \end{bmatrix}$$

The spectrum is

$$\sigma = \{4,\ 0,\ (-2)^2,\ (\sqrt{2})^2,\ (-\sqrt{2})^2\}.$$

The graph is not integral!

For a Cayley graph:

- The degree of regularity is

$$k(S) = \left| \left\{ \{s, s^{-1}\} \mid s, s^{-1} \in S \right\} \right|.$$

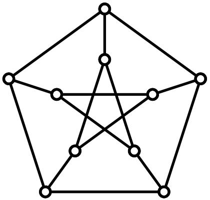
  the number of inverse pairs of elements in S (including self-inverse elements).

- If $\Gamma = \mathrm{Cay}(G, S)$ is bipartite, the eigenvalues of Γ are symmetric in the interval

$$\left[-k(S),\ k(S) \right].$$

- The multiplicity of $k(S)$ is one if the graph is regular and connected.

Petersen's graph:



The spectrum is:

$$\sigma = \{3^1,\ 1^5,\ (-2)^4\}.$$

This graph is integral. But is it a Cayley graph?

Theorem (Sabidussi)

Consider the Cayley graph $\Gamma = (G, S)$ and let $A = \mathrm{Aut}(\Gamma)$ be its automorphism group. Then

$$G \simeq G_L \leq \mathrm{Aut}(\Gamma)$$

acting vertex regularly. Conversely, every graph Γ which has a subgroup $H \leq \mathrm{Aut}(\Gamma)$ acting vertex regularly is a Cayley graph.

Note: Vertex regular implies vertex transitive.

Hence Cayley graphs are vertex transitive.

The converse in not true.

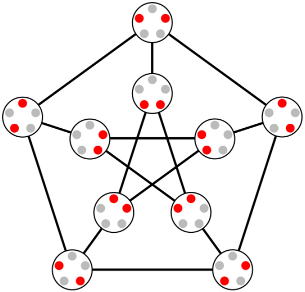
Petersen's graph is vertex transitive. So, is it a Cayley graph?

The problem to decide which vertex transitive graphs are Cayley was originally posed by Marušič in 1983.

To decide if a graph Γ is Cayley, we can compute the automorphism group $\mathrm{Aut}(\Gamma)$ and then find a regular subgroup $H \leq \mathrm{Aut}(\Gamma)$.

For the Petersen graph, this yields the following (the final argument is due to Praeger):

Let us consider the Petersen graph as a Kneser graph $K_{5,2}$:



Vertices correspond to the 2-subsets of $\{1,...,5\}$.
An edge exists if the subsets are disjoint.
$\mathrm{Aut}(\Gamma) = \mathrm{Sym}_5$ is vertex transitive.

A vertex regular group would have order 10.
It would contain an element of order 2. Such an element of order 2 would also have to act regularly (i.e., have no fixpoint on vertices).

A transposition $(a,b)(c)(d)(e)$ fixes 4 vertices, namely

$$\{a,b\},\ \{c,d\},\ \{c,e\},\ \{d,e\}.$$

A double transposition $(a,b)(c,d)(e)$ fixes the two vertices

$$\{a,b\},\ \{c,d\}.$$

Since every involution has fixpoints, a regular subgroup of order 10 cannot be found.
This means that Petersen's graph is not a Cayley graph.

Recall two types of groups:

1. The symmetric group $\mathrm{Sym}(X)$.
We write Sym_n if $X = \{1, 2, \ldots, n\}$.
We have two notations for elements in $\mathrm{Sym}(X)$. The two notations

$$\begin{bmatrix} 1\,2\,3\,4\,5 \\ 4\,5\,2\,1\,3 \end{bmatrix}$$

and $(1,4)(2,5,3)$ describe the same element in Sym_5.

It is the mapping

$$\begin{aligned} 1 &\mapsto 4, \\ 2 &\mapsto 5, \\ 3 &\mapsto 2, \\ 4 &\mapsto 1, \\ 5 &\mapsto 3. \end{aligned}$$

The first notation is called *list notation.* The second notation is called *cycle-notation.*

2. Linear groups such as

$$\mathrm{PGL}(n, q)$$

This is the group of $n \times n$ matrices over the finite field $\mathbb{F}_q$, acting on the projective space $\mathrm{PG}(n-1, q)$. This just means we consider the induced action of $\mathrm{GL}(n, q)$ on the set of one-dimensional subspaces of $\mathbb{F}_q^n$ (which form a system of imprimitivity).

Example (Lubotzky, Phillips and Sarnak 1988)

Pick two distinct prime numbers p and q, both congruent to 1 mod 4. Let $G = \mathrm{PGL}(2, q)$ a group of order $(q+1)q(q-1)$. Let

$$S = \left\{ \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix} \right\},$$

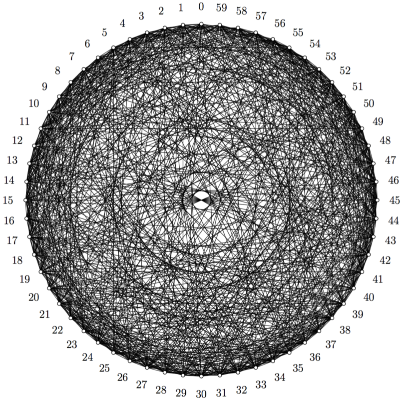
in $\mathrm{PGL}(2,q)$, where $a^2+b^2+c^2+d^2 = p$ and $i^2 \equiv -1$ mod q, $i \in \mathbb{F}_p$.

The Cayley graph

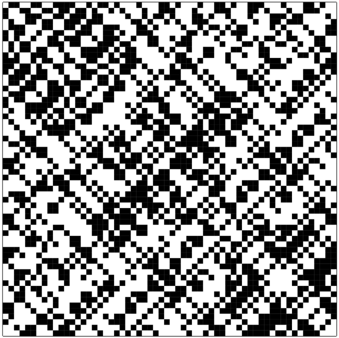
$$X^{p,q} = \mathrm{Cay}(G, S)$$

is an example of a class of graphs called expander graphs. For $p = 29, q = 5$ we get two disconnected (but isomorphic) graphs on 60 vertices.

Here is one half of $X^{29,5}$:



Here is the adjacency matrix:



The spectrum is integral:

$$\sigma = \{27^1,\ (-3)^4,\ (-7)^{12},\ (-1)^{15},\ 3^{28}\}.$$

The Laplace spectrum is

$$\Lambda = \{0,\ 24^{28},\ 28^{15},\ 30^4,\ 34^{12}\}.$$

So, $\mu_2 = 24$.

Two more Cayley graphs can be defined:

Definition (Bubble Sort Graph)

$\Gamma_n := \mathrm{Cay}(\mathrm{Sym}_n, S_n)$ where

$$S_n = \{(1,2),\ (2,3),\ \dots,\ (n-1,n)\}.$$

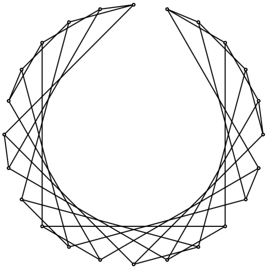
Definition (Star Graph)

$\mathrm{Star}_n := \mathrm{Cay}(\mathrm{Sym}_n, T_n)$ where

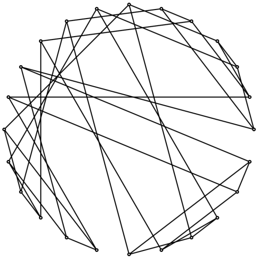
$$T_n = \{(1,2),\ (1,3),\ \dots,\ (1,n)\}.$$

Which graph is nicer?

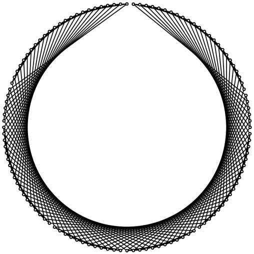
$\Gamma_4 =$



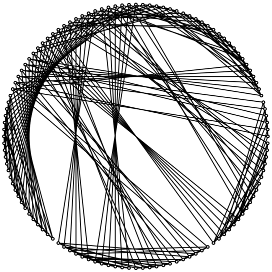
$\text{Star}_4 =$



$\Gamma_5 =$



$\text{Star}_5 =$



It depends, but in terms of integrality, the star graph wins.

The Bubble sort graphs are not integral, the star graphs are (see below).

$$\sigma(\Gamma_4) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$$

but

$$\sigma(\mathrm{Star}_4) = \left\{(-3)^1,\ (-2)^6,\ (-1)^3,\ 0^4,\ 1^3,\ 2^6, 3^1\right\}.$$

In 2009, Abdollahi and Vatandoost ask

1. Is Star_n integral?
2. Does every integer in the interval $[-(n-1),(n-1)]$ appear in the spectrum of Star_n?

In 2012, Krakovski and Mohar confirm the second part of the conjecture.

Theorem (Krakovski and Mohar, 2012)

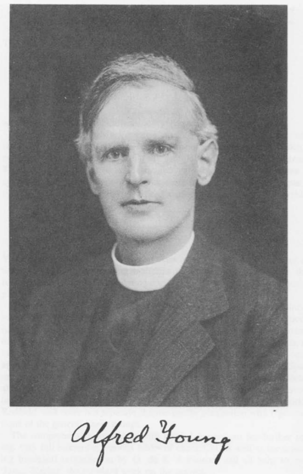
For each integer $1 \leq \ell \leq n-1$, the integers $\pm(n-\ell)$ are eigenvaues of Star_n with multiplicity at least $\binom{n-2}{\ell-1}$. If $n \geq 4$, then 0 is an eigenvalue of Star_n with multiplicity at least $\binom{n-1}{2}$.

The multiplicity bounds in this result are not strong enough to show that *every* eigenvalue is an integer.

Strangely enough, the first question was answered potentially up to 100 years *before it was posed.*

Alfred Young in a series of papers published in the 1890's-1900's contributed to the representation theory of the symmetric group.

This is key to answering this question in the positive.



Alfred Young (1873-1940)

We know exactly which integers in

$$\Big[-(n-1),(n-1)\Big]$$

appear as eigenvalues of Star_n and what the multiplicity is.

The answer relies on the theory of representations of the group Sym_n.

Definition

A *representation* of a group G is a homomorphism

$$G \to \mathrm{GL}(V)$$

for some vector space V.

For the purposes of this talk, the vector space V is finite dimensional complex.

We can narrow down V to be the group ring $\mathbb{C}[G]$.

Elements are the formal sums (linear combinations) of the form

$$\sum_{g \in G} a_g g$$

where $a_g \in \mathbb{C}$ for all $g \in G$.

- Addition is component-wise
- Scalar multiplication
- Multiplication is convolution

The group algebra is a vector space of dimension $|G|$ over $\mathbb{C}$.

A basis is given by the elements of G (thought of as members of $\mathbb{C}[G]$).

Let us call this the *standard basis.*

Multiplication by a group element h on the right is a linear map:

$$h_R : \mathbb{C}[G] \to \mathbb{C}[G], \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g gh.$$

This gives us a group

$$G_R = \{h_R \mid h \in G\}$$

called the right regular representation.

With respect to the standard basis of $\mathbb{C}[G]$, we have representations

$$G \to \mathrm{GL}(|G|, \mathbb{C}), \quad h \mapsto \mathrm{Mat}(h_R)$$

where $\mathrm{Mat}(h_R)$ is the matrix associated with h_R.

The matrix $\mathrm{Mat}(h_R)$ is a permutation matrix.

Example

Let $G = \mathbb{Z}_8 = \{0, \ldots, 7\}$.

$$\mathrm{Mat}(1_R) = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0 \end{bmatrix}$$

We can lift the representation to the group algebra

Example

Let $G = \mathbb{Z}_8 = \{0, \ldots, 7\}$.

$$\mathrm{Mat}(1_R + (-1)_R + 2_R + (-2)_R) = \begin{bmatrix} 0&1&1&0&0&0&1&1\\ 1&0&1&1&0&0&0&1\\ 1&1&0&1&1&0&0&0\\ 0&1&1&0&1&1&0&0\\ 0&0&1&1&0&1&1&0\\ 0&0&0&1&1&0&1&1\\ 1&0&0&0&1&1&0&1\\ 1&1&0&0&0&1&1&0 \end{bmatrix}.$$

The main idea is to realize that the adjacency matrix of the star graph is the regular representation of the connection set T_n.

Example

Star_3 has adjacency matrix

$$\mathrm{Mat}\Big((1,2)_R + (1,3)_R\Big) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues can be computed using the Young tableau of order n.

Let us explain.

First, a representation (such as the regular representation G_R) may or may not be irreducible.

A representation is *reducible* if there is a submodule which is mapped to itself under every representation. So,

$$G \to \mathrm{GL}(V),\ g \mapsto \begin{bmatrix} A_g & 0 \\ C_g & B_g \end{bmatrix}.$$

A submodule can be complemented by another submodule, and hence the representation can be reduced to block diagonal form

$$G \to \mathrm{GL}(V),\ g \mapsto \begin{bmatrix} A_g & 0 \\ 0 & B_g \end{bmatrix}.$$

The maps $g \mapsto A_g$ and $g \mapsto B_g$ are smaller representations.

Continuing to chop the representation into smaller ones, we obtain the *irreducible representations* of G.

The module $V = \mathbb{C}[G]$ is semi-simple and hence can be reduced to a direct sum of irreducible modules.

There is a notion of equivalence of representations (details omitted).

What one learns is that the regular representation of a group G splits into the direct sum of the irreducibles, and the following quantities are equal:

- The dimension f_i of a particular irreducible representation afforded by the module M_i.
- The multiplicity of this representation as a constituent in the regular representation.

If we denote this multiplicity by f_i then

$$G = \bigoplus_{i=1}^{h} f_i M_i, \qquad |G| = \sum_{i=1}^{h} f_i^2.$$

Here, h is the number of inequivalent irreducible representations of G.

It is known that h is the number of conjugacy classes of G.

For the symmetric group, partitions are important:

Fix a partition $\alpha = (a_1, \ldots, a_k)$ of n, so that $n = a_1 + \cdots + a_k$ and $a_1 \geq a_2 \geq \cdots \geq a_k$.

The *diagram* of α is the set

$$D_\alpha = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq k,\ 1 \leq j \leq a_i\}.$$

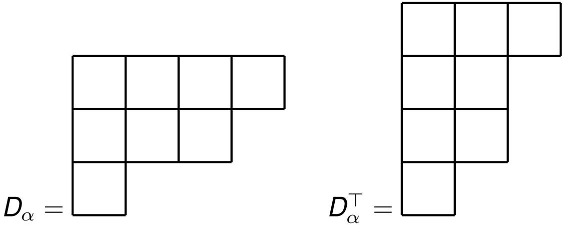
EXAMPLE:
We use the English notation, by which the index i goes down the rows and the index j goes across the columns.

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	
(3,1)			

The transposed diagram $D_\alpha^\top$ is

$$D_\alpha^\top = \{(j, i) \in \mathbb{N} \times \mathbb{N} \mid (i, j) \in D_\alpha\}.$$

EXAMPLE:
Here is a diagram and its transpose.



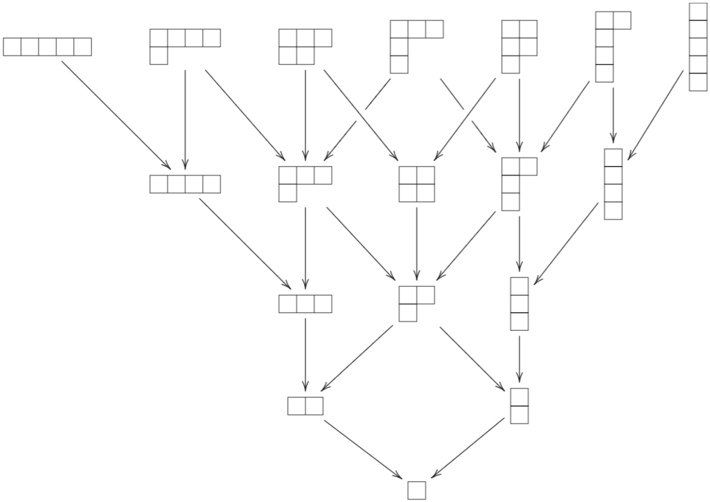
The *conjugate partition* of α is the partition α' such that

$$D_{\alpha'} = D_\alpha^\top.$$

Thus, in the example:

$$(4, 3, 1)' = (3, 2, 2, 1).$$

The Young lattice:



We need to consider paths in this lattice.

Each path is recorded as a *Young tableau.*

A *Young tableau* associated with a partition α is a map

$$y: \; D_\alpha \to [1, n]$$

such that

$$y(i,j) > y(i,j-1) \quad \text{for all } (i,j) \in D_\alpha, \; j > 1$$

and

$$y(i,j) > y(i-1,j) \quad \text{for all } (i,j) \in D_\alpha, \; i > 1.$$

These conditions imply that the mapping is a bijection. It is common practice to represent the mapping y by listing the image values in the appropriate box in the diagram.

EXAMPLE:
Here is an example of a Young tableau for the partition $\alpha = (5, 2)$

1	3	5
2	4	

There are exactly 5 Young tableaux for the partition $(3, 2)$:

1	2	3
4	5	

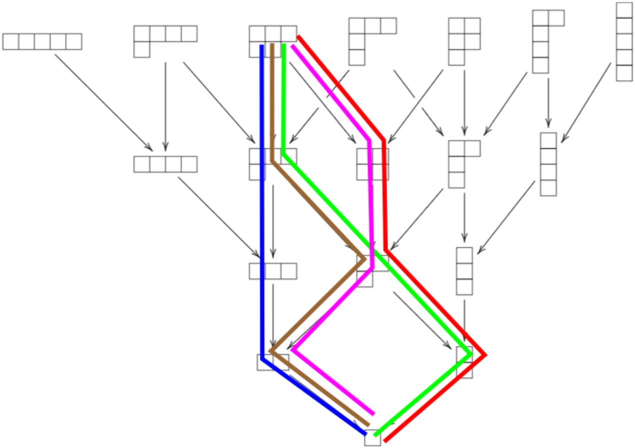
1	2	4
3	5	

1	2	5
3	4	

1	3	4
2	5	

1	3	5
2	4	

These 5 tableaux correspond to 5 paths in the young lattice:



To every Young tableau y, we define two groups:

1. The row-group is the group $\mathcal{R}_y$ of permutations which move elements within the rows of y only (also called the row-stabilizer).

2. The column-group is the group $\mathcal{C}_y$ of permutations which move elements within the columns of y only (also called the column-stabilizer).

In the example,

1	3	5
2	4	

we have

$$\mathcal{R}_y = \langle (1,3),(1,5),(2,4)\rangle \simeq \mathrm{Sym}(\{1,3,5\}) \times \mathrm{Sym}(\{2,4\})$$

a group of order 12.

The column group is

$$\mathcal{C}_y = \langle (1,2),(3,4)\rangle \simeq C_2 \times C_2$$

a group of order 4.

The *Young symmetrizer* w_y is defined as

$$w_y = \sum_{g \in \mathcal{R}_y} \sum_{h \in \mathcal{C}_y} \mathrm{sgn}(h) g h.$$

Theorem (Young)

The span of w_y, i.e. the left ideal $\mathbb{C}[\mathrm{Sym}_n] \cdot w_y$ is an irreducible module.

Two modules w_y and $w_{y'}$ are isomorphic if and only if y and y' have the same shape α. Any given isomorphism type of module appears with a certain multiplicity f_α.

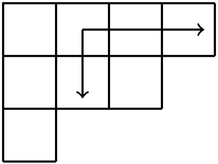
The dimension f_α is also the number of ways that a tableau can be defined for a fixed shape α.

There is a formula which allows to compute f_α. This is the “hook formula.”

Start with a partition

$$\alpha = (a_1, \ldots, a_k), \quad a_1 \geq a_2 \geq \cdots a_k.$$

The *hook* associated to a node (i, j) ($1 \leq i \leq s$, $1 \leq j \leq a_i$) is all nodes weakly to the right and weakly below that node in the diagram D_α.



Here, we have hook length $h_{1,2} = 4$.

The *hook length* of a pair $(i, j) \in D_\alpha$ can be computed as

$$h_{i,j} = a_i - j + a'_j - i + 1,$$

where $\alpha' = (a'_1, \ldots, a'_{a_1})$ is the conjugate partition of α.

Theorem (Frame, Robinson, Thrall 1954)

Let $\alpha = (a_1, \ldots, a_k)$ *be a partition of* n*. Let* $\alpha' = (a'_1, \ldots, a'_{a_1})$ *be the conjugate partition. Then*

$$f_\alpha = \frac{n!}{\prod_{i=1}^{k} \prod_{j=1}^{a_i} h_{ij}}$$

where

$$h_{ij} = a_i - j + a'_j - i + 1.$$

Here are the hook lengths of the diagram from above:

4	3	1
2	1	

This gives

$$f_{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 2} = 5.$$

EXAMPLE:
For Sym_4, we have the partitions

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

The tableau are

$$y_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$$

$$y_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad y_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad y_4 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$y_5 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \quad y_6 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

$$y_7 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad y_8 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad y_9 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \quad y_{10} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$$

The dimensions of the irreducible representations by partition are

$$f_{(4)} = 1,\ f_{(3,1)} = 3,\ f_{(2,2)} = 2,\ f_{(2,1,1)} = 3,\ f_{(1,1,1,1)} = 1.$$

So, the dimension formula becomes

$$24 = 1^2 + 3^2 + 2^2 + 3^2 + 1^2.$$

The Young symmetrizers are

i	w_{y_i}
1	$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$
2	$(1,0,1,0,0,0,1,0,1,-1,0,-1,1,0,1,-1,0,-1,0,0,0,-1,0,-1)$
3	$(1,0,0,0,0,1,1,0,-1,0,-1,1,0,0,-1,0,-1,0,0,1,-1,1,-1,0)$
4	$(1,1,0,0,0,0,-1,-1,0,0,0,0,-1,-1,1,1,0,0,-1,-1,1,1,0,0)$
5	$(1,1,0,0,-1,-1,1,1,-1,-1,0,0,0,0,-1,-1,1,1,-1,-1,0,0,1,1)$
6	$(1,-1,0,-1,0,1,-1,1,0,1,0,-1,-1,0,1,0,1,-1,1,0,-1,0,-1,1)$
7	$(1,-1,0,0,0,0,1,-1,-1,1,1,-1,0,0,-1,1,0,0,0,0,1,-1,0,0)$
8	$(1,0,0,0,0,-1,-1,0,0,0,0,1,-1,1,1,-1,-1,1,0,1,0,-1,0,0)$
9	$(1,0,-1,0,0,0,-1,0,1,0,0,0,1,0,-1,0,0,0,1,-1,-1,1,1,-1)$
10	$(1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1)$

If we span each module and concatenate bases, we get the following 24×24 matrix B:

$$
\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline\hline
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & -1 \\
\hline
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
\hline\hline
1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
\hline
1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 \\
0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & 1 & 0 \\
\hline\hline
1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
\hline
1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\
\hline
1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
\hline\hline
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1
\end{array}\right]
$$

Computing

$$B \cdot \mathcal{A}_4 \cdot B^{-1}$$

yields the block diagonal matrix

$$\begin{bmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{bmatrix}$$

Because isomorphic modules yield the same eigenvalues, we only need to look at one block for each partition.

The blocks associated to the partitions are

$$\begin{bmatrix} 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -3 \end{bmatrix}.$$

The set of eigenvalues which we get from these matrices are (with block multiplicities as exponents)

$$\Big\{3\Big\}^1, \Big\{(-1), 2^2\Big\}^3, \Big\{0^2\Big\}^2, \Big\{1, (-2)^2\Big\}^3, \Big\{-3\Big\}^1.$$

Collecting these sets, we determine the spectrum of Star_4 as

$$\sigma = \Big\{3,\ (-1)^3,\ 2^6,\ 0^4,\ 1^3,\ (-2)^6,\ -3\Big\}.$$

The story continues ...

It is possible to determine the exact eigenvalues using the Young tableaux:

For a fixed partition α, the eigenvalue d appears in the irreducible representation of type α in $m_{\alpha,d}$ ways.

Here, $m_{\alpha,d}$ is the number of Young tableaux of shape α with the number n in the d-th diagonal.

This completely settles the problem.

To prove this result, the Young-Jucys-Murphy elements in $\mathbb{C}[\mathrm{Sym}_n]$ need to be considered.

The Young-Jucys-Murphy elements are the expressions

$$y_k = (1, k+1) + (2, k+1) + \cdots + (k, k+1).$$

Obviously, the representing matrix of

$$(y_{n-1})_R$$

is the adjacency matrix of the Star graph (after rearranging vertices according to the permutation $1 \leftrightarrow n$).

The details in this connection are buried in the new approach to the representation theory of the symmetric group as proposed by Okounkov and Vershik.

In this approach, the Young-Yucys-Murphy elements play an important role (details omitted).

Our own contribution:

1. A comment on $\mathrm{Aut}(\mathrm{Star}_n)$.
2. A comment on the adjacency matrix of Star_n.
3. A comment on $\mathrm{Aut}(\Gamma_n)$.

Lemma

$$(\mathrm{Sym}_n)_R.\mathrm{Inn}\Big(\mathrm{Sym}_{1,n-1}\Big) \leq \mathrm{Aut}(\mathrm{Star}_n).$$

The group acts imprimitively on the vertices.
The partition $\mathcal{C} = (C_1, \ldots, C_n)$ *with* $C_i = \{g \in \mathrm{Sym}_n \mid 1^g = i\}$ *is an invariant partition.*
For $b \in \mathrm{Inn}(\mathrm{Sym}_{1,n-1})$

$$C_1^b = C_1, \text{ and } C_i^b = C_{i^b} \text{ for } i = 2, \ldots, n.$$

Proof: It is clear that $A = \mathrm{Sym}_n$ acts in the right regular representation on Star_n.

Let $B = \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Sym}_n)}(T_n)$.

Then $B = \mathrm{Sym}_{n-1} \leq \mathrm{Inn}(\mathrm{Sym}_n)$ (even for $n = 6$, when $\mathrm{Aut}(\mathrm{Sym}_6)$ is larger than Sym_6).

So, for every edge (g, sg), the image under $\alpha \in B$ is

$$(g^\alpha, (sg)^\alpha) = (g^\alpha, s^\alpha g^\alpha) = (h, s'h)$$

for $h = g^\alpha$ and $s' = s^\alpha \in S$.

The action of B on the vertices of Star_n is given by the conjugation action on the underlying group elements in Sym_n.

This leaves one question: Is it true that

$$\mathrm{Aut}(\mathrm{Star}_n) = \mathrm{Sym}_n.\mathrm{Sym}_{n-1} \text{ ?}$$

I have verified this for all $n \leq 6$.

Lemma

Consider the family of graphs $\mathrm{Star}_n := \mathrm{Cay}(\mathrm{Sym}_n, T_n)$. *Order the vertices lexicographically. Partition the adjacency matrix* $\mathcal{A}_n$ *according to the classes* $C_i = \{\pi \in \mathrm{Sym}_n \mid 1^\pi = i\}$. *Then* $\mathcal{A}_n$ *has the block matrix form*

$$\mathcal{A}_n = \begin{bmatrix} 0 & P_2^\top & P_3^\top & \cdots & P_n^\top \\ P_2 & \mathcal{A}_{n-1} & 0 & \cdots & 0 \\ P_3 & 0 & \mathcal{A}_{n-1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ P_n & 0 & \cdots & 0 & \mathcal{A}_{n-1} \end{bmatrix} \tag{1}$$

where each submatrix is of size $(n-1)! \times (n-1)!$.
P_k *is the permutation matrix of the mapping*

$$\begin{aligned} \psi_k : \quad & C_k \to C_1, \\ \sigma \quad & \mapsto \sigma^{\psi_k}, \text{ where } i^{\sigma^{\psi_k}} := \begin{cases} i & \text{if } i \neq 1 \\ k & \text{if } i = 1 \end{cases} \end{aligned}$$

The Lemma can be illustrated:

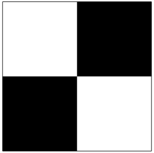
Example: Here is the lexicographic ordering of elements of Sym_4

0	1	2	3	4
1	1	2	4	3
2	1	3	2	4
3	1	3	4	2
4	1	4	2	3
5	1	4	3	2
6	2	1	3	4
7	2	1	4	3
8	2	3	1	4
9	2	3	4	1
10	2	4	1	3
11	2	4	3	1

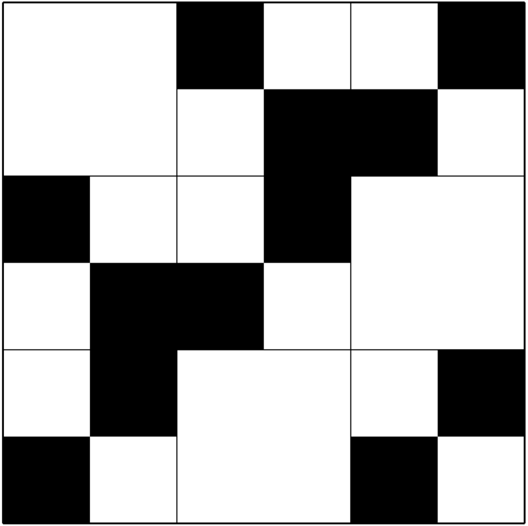
12	3	1	2	4
13	3	1	4	2
14	3	2	1	4
15	3	2	4	1
16	3	4	1	2
17	3	4	2	1
18	4	1	2	3
19	4	1	3	2
20	4	2	1	3
21	4	2	3	1
22	4	3	1	2
23	4	3	2	1

Note that the lexicographic order is compatible with the partition.

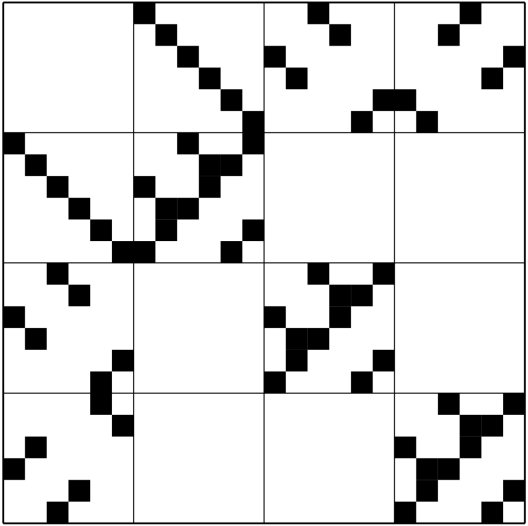
Star_2 :



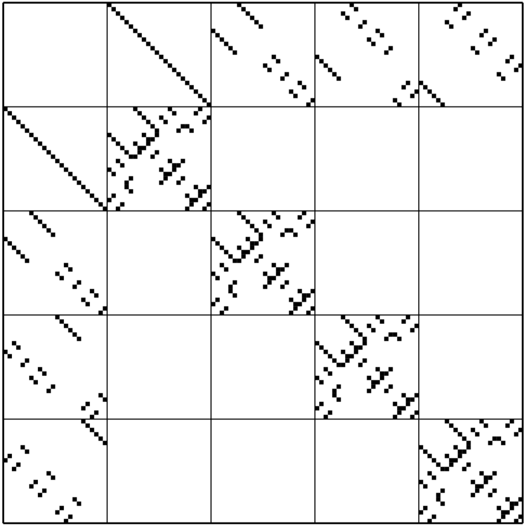
Star$_3$:



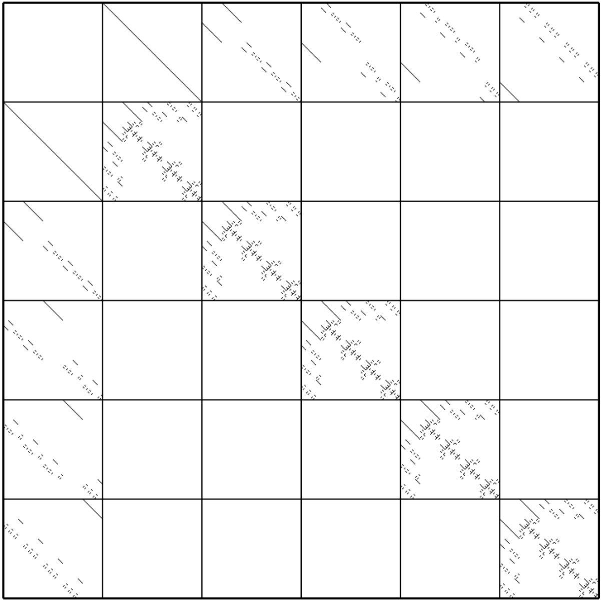
Star$_4$:



Star$_5$:



Star$_6$:



And now for the Bubble Sort graph Γ_n:

Lemma

Let

$$\iota = \begin{bmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{bmatrix}.$$

Then

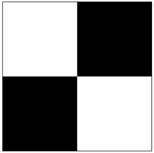
$$(\mathrm{Sym}_n)_R.\mathrm{Inn}\Big(\langle \iota \rangle\Big) \leq \mathrm{Aut}(\Gamma_n).$$

This leaves one question: Is it true that

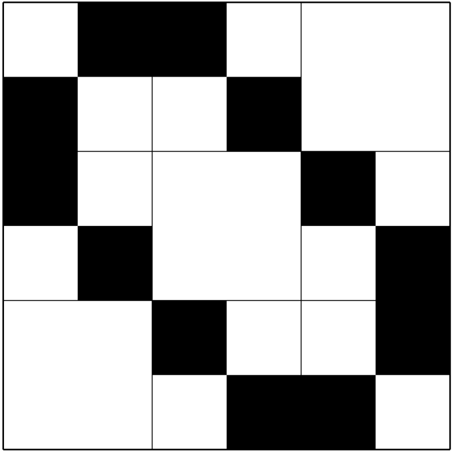
$$\mathrm{Aut}(\Gamma_n) \simeq \mathrm{Sym}_n.\mathrm{Sym}_2 \text{ ?}$$

I have verified this for all $n \leq 6$.

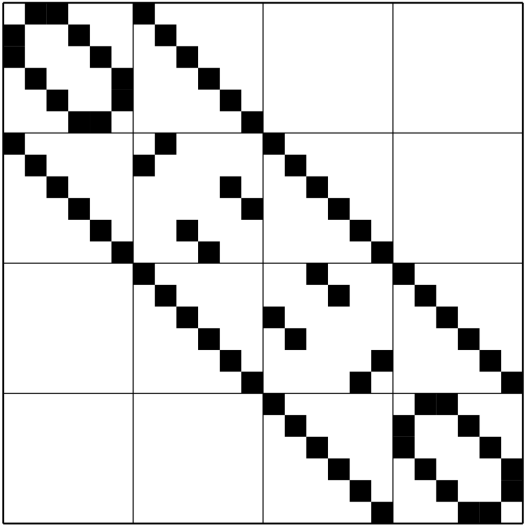
Γ_2 :



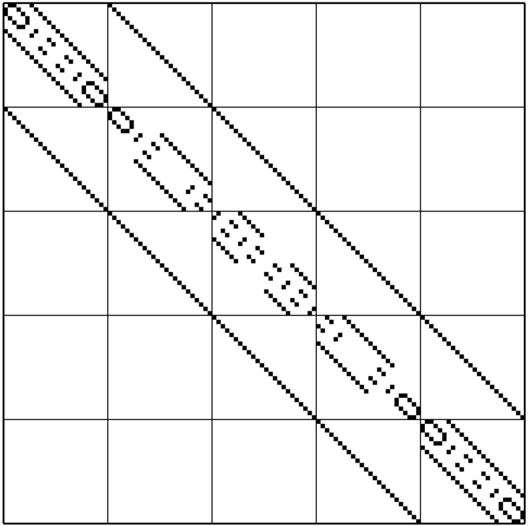
Γ_3 :



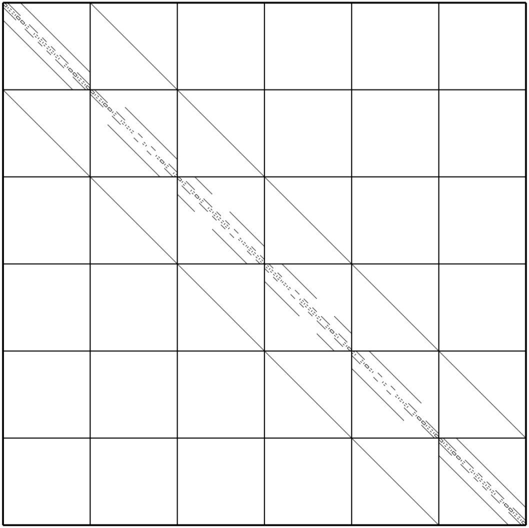
Γ_4 :



Γ_5 :



Γ_6 :



Thank You All!