A construction of infinite families of directed strongly regular graphs

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Introduction

Though this is not joint work with anybody, all the tricks used here I learned from Misha Klin.

Strongly regular graphs

Definition 1.

A regular graph $\Gamma = (V, E)$ of order n and degree k is called strongly regular with parameters (n, k, λ, μ), if it is neither complete nor edgeless, and there are integers λ and μ such that:

- Every two adjacent vertices have λ common neighbours.
- Every two non-adjacent vertices have μ common neighbours.

Strongly regular graphs

Definition 2.

A simple graph $\Gamma = (V, E)$ of order n is called strongly regular with parameters (n, k, λ, μ), if it is neither complete nor empty, and there exist constants k, λ, μ such that for any $u, v \in V$ the number of uv-walks of length 2 is

$$\begin{cases} k, & \text{if } u = v, \\ \lambda, & \text{if } uv \in E, \\ \mu, & \text{if } uv \notin E. \end{cases}$$

Remark.

In this fashion we will define directed SRGs.

Strongly regular graphs

Let $A = A(\Gamma)$ denote the adjacency matrix of Γ. Then

$$A^2 = k \cdot I + \lambda \cdot A + \mu \cdot (J - I - A),$$

or equivalently,

$$A^2 + (\mu - \lambda) \cdot A - (k - \mu) \cdot I = \mu \cdot J,$$

where I is the identity matrix and J the all-one matrix.

Directed strongly regular graphs

Definition (Duval, 1988)

Let $\Gamma = (V, D)$ be a directed graph, $|V| = n$, in which vertices have constant in- and out-valency k, but now only t edges being undirected $(0 < t < k)$. We say that Γ is a directed strongly regular graph with parameters (n, k, t, λ, μ) if there exist constants λ and μ such that the numbers of uw-paths of length 2 are

1. t, if $u = w$;
2. λ, if $(u, w) \in D$;
3. μ, if $(u, w) \notin D$.

$$A^2 = tI + \lambda A + \mu(J - I - A).$$

Directed strongly regular graphs

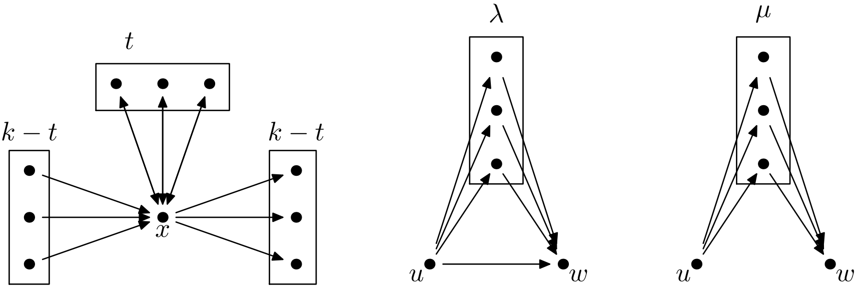

Figure: Locally.

Directed strongly regular graphs

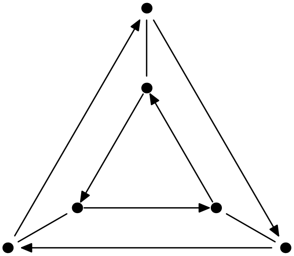

Figure: The smallest DSRG.

The parameter set is $(6, 2, 1, 0, 1)$.

Directed strongly regular graphs

Proposition (Duval, 1988)

If Γ is a DSRG with parameter set (n, k, t, λ, μ) and adjacency matrix A, then the complementary graph $\bar{\Gamma}$ is a DSRG with parameter set $(n, \bar{k}, \bar{t}, \bar{\lambda}, \bar{\mu})$ with adjacency matrix $\bar{A} = J - I - A$, where

$$\begin{aligned} \bar{k} &= n - k - 1 \\ \bar{t} &= n - 2k + t - 1 \\ \bar{\lambda} &= n - 2k + \mu - 2 \\ \bar{\mu} &= n - 2k + \lambda. \end{aligned}$$

Directed strongly regular graphs

Proposition (Ch. Pech, 1997)

Let Γ be a DSRG. Then its reverse Γ^T is also a DSRG with the same parameter set.

Definition

We say that two DSRGs Γ_1 and Γ_2 are equivalent, if $\Gamma_1 \cong \Gamma_2$, or $\Gamma_1 \cong \Gamma_2^T$, or $\Gamma_1 \cong \bar{\Gamma}_2$, or $\Gamma_1 \cong \bar{\Gamma}_2^T$; otherwise they are called non-equivalent.

Directed strongly regular graphs

Duval's main theorem

Let Γ be a DSRG with parameters (n, k, t, λ, μ). Then there exists some positive integer d for which the following requirements are satisfied:

$$k(k + (\mu - \lambda)) = t + (n - 1)\mu$$

$$(\mu - \lambda)^2 + 4(t - \mu) = d^2$$

$$d \mid (2k - (\mu - \lambda)(n - 1))$$

$$\frac{2k - (\mu - \lambda)(n - 1)}{d} \equiv n - 1 \pmod{2}$$

$$\left| \frac{2k - (\mu - \lambda)(n - 1)}{d} \right| \leq n - 1.$$

Directed strongly regular graphs

Further necessary conditions

$$\begin{array}{rcccl} 0 & \leq & \lambda < t & < & k \\ 0 & < & \mu \leq t & < & k \\ -2(k-t-1) & \leq & \mu - \lambda & \leq & 2(k-t). \end{array}$$

Directed strongly regular graphs

Usually, the main goals concerning DSRG's are:

1. To find a DSRG realizing a "new" parameter set.
2. To prove a non-existence result.
3. To find an infinite family of DSRG's.

The most important data are collected on the webpage of A. Brouwer and S. Hobart: http://homepages.cwi.nl/~aeb/math/dsrg

The π-join construction

- Γ – (D)SRG of order n;
- $\pi = \{C_1, C_2, \ldots, C_a\}$ – a homogeneous partition $V(\Gamma)$ with a cells of size b.
- A adjacency matrix of Γ respecting π;
- $U_i = (0, \ldots, 0, 1, 0, \ldots, 0) \otimes J$;
- j – any positive integer;
- $M_j(A)$ - circulant block matrix whose first row is

$$(A, \underbrace{U_1, \ldots, U_1}_{j}, \underbrace{U_2, \ldots, U_2}_{j}, \ldots, \underbrace{U_a, \ldots, U_a}_{j}).$$

- The j-th π-join of Γ is a digraph with adjacency matrix $M_j(A)$.

Example $K_{2,2}$ with $a = b = j = 2$

$ja + 1 = 5$ copies of $K_{2,2}$:

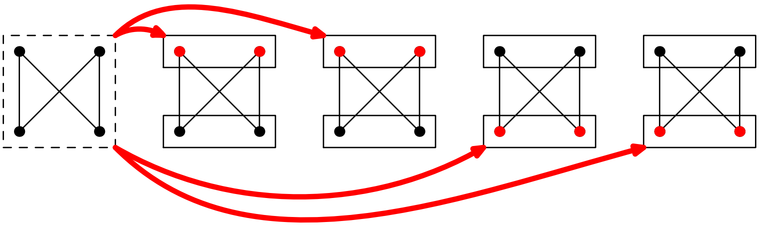

Figure: π-join of $K_{2,2}$.

Example with DSRGs

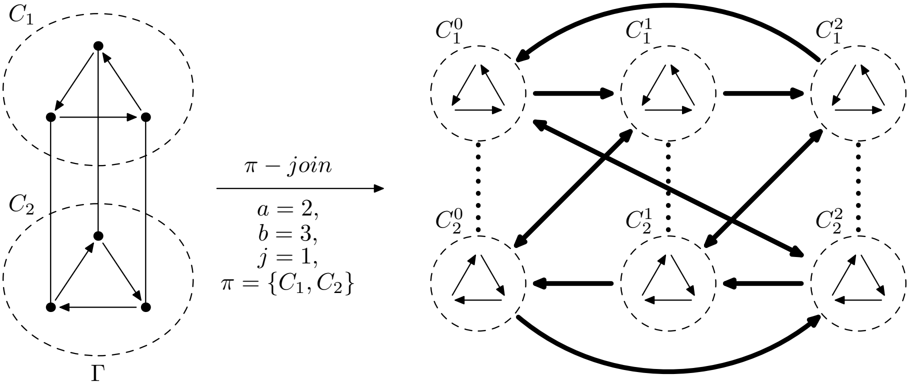

Figure: π-join of DSRG(6,2,1,0,1) is a DSRG(18,8,4,3,4).

Motivation

A significant amount of known DSRGs can be constructed using π-join construction from smaller (D)SRGs.

Problem.

When does the π-join construction give a new DSRG from a small one?

Equitable partitions and SRGs

Key tool

It turned out that special equitable partitions play the key role here.

Equitable partitions have been recently applied in the theory of undirected SRGs by Hirasaka, Kang, Kim (2006), and in the dissertation thesis of M. Ziv-Av (2014).

Equitable partitions

A partition $\pi = \{C_1, \ldots, C_r\}$ of the vertex set of a digraph is called out-equitable if for every $i, j \in \{1, \ldots, r\}$ the number $q^+_{i,j} = |N^+(u) \cap C_j|$ does not depend on the concrete choice of $u \in C_i$, just on the indices i and j.
Similarly, a partition $\pi = \{C_1, \ldots, C_r\}$ of the vertex set is called in-equitable if for every $i, j \in \{1, \ldots, r\}$ the number $q^-_{i,j} = |C_i \cap N^-(v)|$ does not depend on the concrete choice of $v \in C_j$, just on the indices i and j.
The corresponding matrix $Q = (q_{i,j})$ is called quotient matrix.

Necessary conditions

Observation

Let a, b, k, A, U_i are as in the definition of the π-join construction. Then for any $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, b\}$ we have:

(i) $\sum_{i=1}^{a} U_i = J$,

(ii) $U_i \cdot U_j = b \cdot U_j$,

(iii) $A \cdot U_j = k \cdot U_j$,

(iv) All the rows in $U_i \cdot A$ are equal, and the entry in an arbitrary column represents the number of darts starting from cell C_i terminating in the vertex corresponding to the given column.

Necessary conditions

Theorem 1.

Let A be an adjacency matrix of a (D)SRG Γ with parameters (n, k, t, λ, μ), which respects the homogeneous partition $\pi = \{C_1, \ldots, C_a\}$ of degree b. Suppose that the π-join Γ_π^1 for $j = 1$ is a DSRG with parameters $(\tilde{n}, \tilde{k}, \tilde{t}, \tilde{\lambda}, \tilde{\mu})$. Then

(a) $\tilde{n} = (a+1)n$, $\tilde{k} = n + k$, $\tilde{t} = b + t$, $\tilde{\lambda} = b + \lambda$, and $\tilde{\mu} = b + \mu$.

(b) For arbitrary $i, l \in \{1, \ldots, a\}$ the number $q_{i,l}$ of darts starting in C_i and terminating in $v \in C_l$ does not depend on the concrete choice of v, just on i and l, i.e. π is an in-equitable partition with quotient matrix $Q = (\lambda + b - k)I + \mu(J - I)$.

Necessary conditions

Proof.

$M_1(A)$ – adjacency matrix of the resulting graph;
$M_1(A)^2$ – is clearly a block-circulant matrix.
$(B_0, B_1, \ldots, B_a)$ – first row of blocks of $M_1(A)^2$.

(a) Based on the Observation:

$$B_0 = A^2 + U_1U_a + U_2U_{a-1} + \ldots + U_aU_1 =$$

$$= (\lambda + b)A + (\mu + b)(J - I - A) + (t + b)I.$$

(b) Follows from computing $B_1, B_2, \ldots$.

Necessary conditions

Corollary 1.

$$2k + \mu - \lambda = a\mu + b.$$

Corollary 2.

$$\lambda + b - k \geq 0.$$

Corollary 3.

If $a = 1$, then $b = n$ and the only good basic graphs are complete multipartite.

Corollary 4.

If $b = 1$, then the initial graph Γ is necessarily complete or empty.

Sufficient conditions

Theorem 2.

Let Γ be a (D)SRG with parameter set (n, k, t, λ, μ). Let a and b are positive integers such that $ab = n$ and there exists a homogeneous in-equitable partition $\pi = \{C_1, \ldots, C_a\}$ of vertices with quotient matrix $Q = (\lambda + b - k)I + \mu(J - I)$. Let A be an adjacency matrix respecting π, and let us define matrix $M_j(A)$ in accordance with our π-join construction for an arbitrary positive integer j. Then $M_j(A)$ is an adjacency matrix of a DSRG with parameter set

$$((ja+1)n, jn+k, jb+t, jb+\lambda, jb+\mu).$$

Algorithm

Algorithm

Theorem 2 provides us an algorithm how to proceed when we are looking for a π-join of a (D)SRG:

Step 1. Take a directed or an undirected SRG Γ;

Step 2. Solve $2k + \mu - \lambda = a\mu + b$ for the parameter set of Γ;

Step 3. For the solutions satisfying $\lambda + b - k \geq 0$ count the quotient matrix and find an in-equitable partition π with this quotient matrix.

Step 4. For π, Γ and an arbitrary integer j create the π-join in power j for Γ according to the construction.

Example $K_{2,2}$ alias SRG(4,2,0,2)

$2k + \mu - \lambda = 2 \cdot 2 + 2 - 0 = 6.$
$2a + b = 6, ab = 4 \implies a = b = 2$ or $a = 1,\ b = 4.$
For $a = b = 2$ the quotient matrix is $Q = 0 \cdot I + 2(J - I)$.
For arbitrary $j \in \mathbb{N}$ the resulting π-join is DSRG with parameter set $(4j + 4, 4j + 2, 2j + 2, 2j, 2j + 2)$.

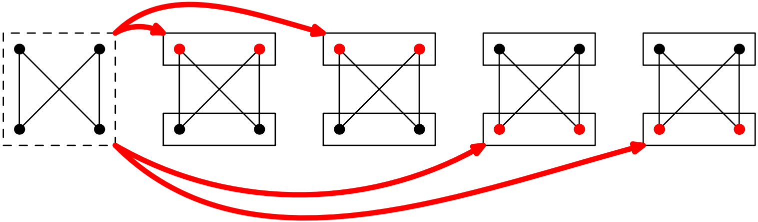

Figure: π-join of $K_{2,2}$.

Results

Main results

- Dozens of infinite families of DSRGs.
- A lot of different well-known constructions can be explained in these terms.
- A lot of new infinite families.
- Up to $n \leq 110$ the number of covered previously open parameter sets is 29.

Further questions

Possible directions of investigation

- Generalisation to non-homogeneous equitable partitions;
- Starting graph is not strongly regular;
- Groups of automorphisms.

References

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Thank you

Thank you for your attention.