Some simple groups which are determined by their character degree graphs

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Summary of this talk

1. Introduction;
2. Definitions;
3. Brief history about character degree graph;
4. Some Lemmas;
5. Main Result;
6. References

Representation, FG-module and character

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- a function

$$\begin{aligned} \theta_\rho : G &\to F \\ g &\mapsto Tr(\rho(g)) \end{aligned}$$

is called a character corresponding to ρ.

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$\mathrm{Irr}(G)$

The set of irreducible characters of G is denoted by $\mathrm{Irr}(G)$.

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$\rho(G)$

The set of prime divisors of the elements of $\mathrm{cd}(G)$ is denoted by $\rho(G)$.

Definitions

$\mathbb{C}G$

Let $\mathbb{C}$ be the complex number field. Then the group algebra of G over $\mathbb{C}$, which is denoted by $\mathbb{C}G$, consists of the set of all sums of the form $\sum_{g\in G} a_g g$, with two multiplications
$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)g$,
$\sum_{g\in G} a_g g \sum_{h\in G} b_h h = \sum_{g\in G} \sum_{h\in G}(a_g a_h)gh$.

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T. Molien proved that $\mathbb{C}G = \oplus_{i=1}^{k} M_{n_i}(\mathbb{C})$, where $n_1, ..., n_k$ are degrees of the irreducible characters of G. Thus for the groups G and H, $X_1(G) = X_1(H)$ if and only if $\mathbb{C}G \cong \mathbb{C}H$.

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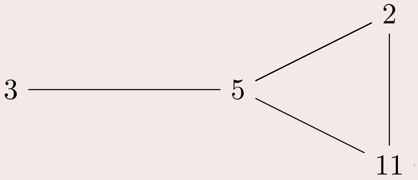
O. Manz, W. Willems, and T. R. Wolf in 1989, introduced a graph based on cd of a group as follows:

$\Delta(G)$

The character degree graph of G, denoted by $\Delta(G)$, is a graph with vertex set $\rho(G)$ and two vertices a and b are adjacent in $\Delta(G)$, if ab divides some irreducible character degree of G.

Example

$\mathrm{cd}(M_{11}) = \{1, 10, 11, 16, 44, 45, 55\}$. Thus $\Delta(M_{11})$ is as follows:



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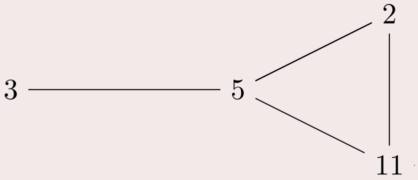

Irreducible constituent

If $\chi = \sum_{i=1}^{N} n_i \chi_i$, where for every $1 \leq i \leq N$, $\chi_i \in \mathrm{Irr}(G)$, then those χ_i with $n_i > 0$ are called irreducible constituents of χ.

Brief history about complex group algebra

In [1], a question (No. 126) has posed as:

Question. Let $X_1(G) = X_1(S_n)$. Is it true $G \cong S_n$? Moreover, if $X_1(G) = X_1(H)$, where H is a simple group, then do we conclude that $G \cong H$?

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For example, $X_1(\mathbb{D}_8) = X_1(\mathbb{Q}_8)$, but $\mathbb{D}_8 \ncong \mathbb{Q}_8$.

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P.P. Palfy, Period. Math. Hung., 1998

Let π be a subset of the vertex set of $\Delta(G)$, with $|\pi| = 3$. Then there is an edge incident to two of the elements of π.

Brief history about character degree graph

D.L. White, J. Algebra, 2008

Let G be a simple group. The graph $\Delta(G)$ is disconnected if and only if $G \cong PSL(2, q)$ for some prime power q. If $\Delta(G)$ is connected, then the diameter of $\Delta(G)$ is at most 3 and $\Delta(G)$ is a complete graph except in the following cases:

1. The diameter of $\Delta(G)$ is 3 if and only if $G \cong J_1$.

2. The diameter of $\Delta(G)$ is 2 if and only if G is isomorphic to one of the following groups:

(a) the Sporadic Mathieu group M_{11} or M_{23},

(b) the alternating group A_8,

(c) the Suzuki group ${}^2B_2(q)$, where $q = 2^{2m+1}$ and $m > 1$,

(d) the linear group $PSL(3, q)$, where $q > 2$ is even or q is odd and $q - 1$ is divisible by a prime other than 2 or 3, or

(e) the unitary group $PSU(3, q)$, where $q > 2$ and $q + 1$ is divisible

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Z. Akhlaghi and H. P. Tong-Viet, Algebr. Represent. Theor, 2015

If G is a finite group such that $\Delta(G)$ is K_4-Free, then $|\rho(G)| \leq 7$.

Brief history about character degree graph

Khosravi's group posed a conjecture as:
Conjecture: Let G and M be two groups such that $\Delta(G) = \Delta(M)$ and $|G| = |H|$. Then $G \cong M$.

B. Khosravi, B. Khosravi, B. Khosravi and Z. Momen

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Ito's Theorem

Let A be an abelian normal subgroup of G. Then $\chi(1) \mid [G : A]$, for all $\chi \in \mathrm{Irr}(G)$.

Some lemmas

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Lemma

Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N, where χ_N is the restriction of χ to N. Then $\frac{\chi(1)}{\theta(1)} \mid [G : N]$.

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Lemma

Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Then for $\beta \in \mathrm{Irr}(\frac{G}{N})$, $\beta\chi \in \mathrm{Irr}(G)$.

Lemma

Let G be a finite solvable group of order $\prod_{i=1}^{n} {p_i}^{a_i}$, where $p_1, p_2, ..., p_n$ are distinct primes. If $kp_n + 1 \nmid {p_i}^{a_i}$ for each $i \leq n-1$ and $k > 0$, then the p_n-Sylow subgroup of G is normal in G.

Main result

Let G be a finite group and let $M \in \{M_{11}, M_{12}, M_{22}, M_{23}\}$. Then $G \cong M$ if and only if $\Delta(G) = \Delta(M)$ and $|G| = |M|$.

Sketch of the proof

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Step 4

L/N is isomorphic to M and so, $G \cong M$.

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THANK YOU