4-COLORED GRAPHS AND COMPLEMENTS OF KNOTS AND LINKS

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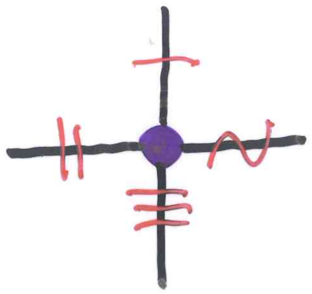
NOVOSIBIRSK 15-28/8/2016

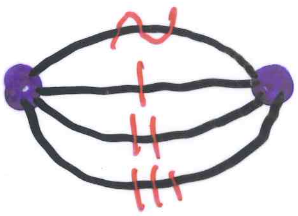
G2S2-2016

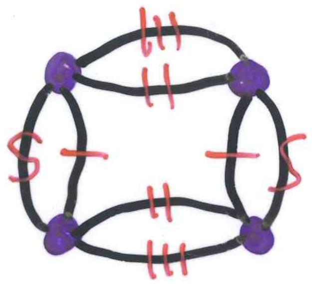
4-COLORED GRAPHS

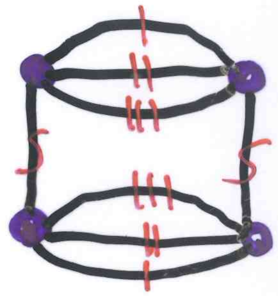


LOCALLY

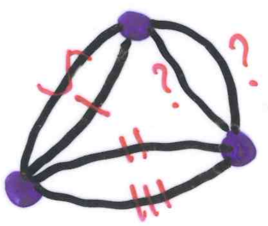
EXAMPLES

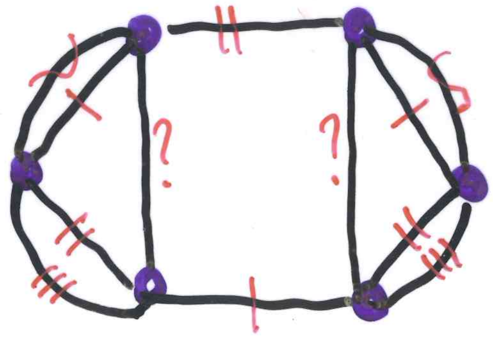






NO-EXAMPLES





RESIDUES

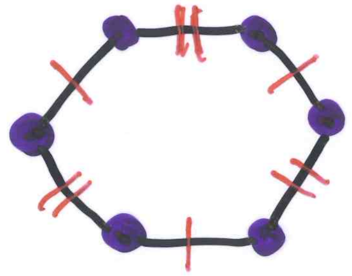
0 - RESIDUES

VERTICES

1- RESIDUES

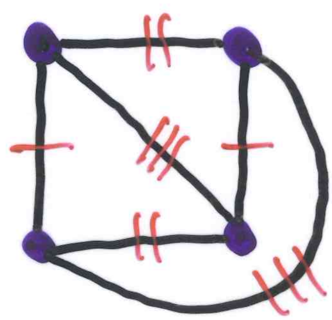
EDGES

2 - RESIDUES



BICOLORED CYCLES

3- RESIDUES



3- GRAPHS ENCODING SURFACES

CONSTRUCTION

1) TAKE THE GRAPH Γ AS 1-SKELETON

2) ATTACH A DISK TO ANY 2-RESIDUE (2-SKELETON)

3') ATTACH A 3-BALL TO ANY 3-RESIDUE WHICH IS S^2 (ORDINARY)

3'') ATTACH $S \times I$ ALONG $S \times \{0\}$ TO ANY 3-RESIDUE WHICH IS $S \neq S^2$ (SINGULAR)

RESULT: A COMPACT 3-MANIFOLD M_Γ WITH (POSSIBLY EMPY) BOUNDARY WITHOUT SPHERICAL COMPONENTS

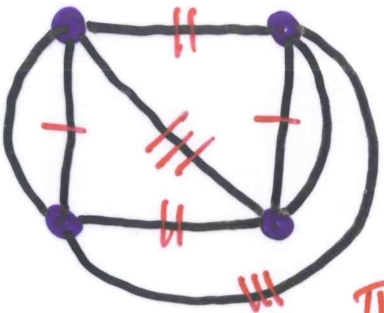
REMARKS: i) IF ALL 3-RESIDUES OF Γ ARE ORDINARY THE MANIFOLD M_Γ IS CLOSED. OTHERWISE IT IS WITH NON-EMPTY BOUNDARY

ii) BOUNDARIES ~~ARE COLORED~~ WITH THE MISSING COLOR OF THE ASSOCIATED ~~3-RESIDUE~~

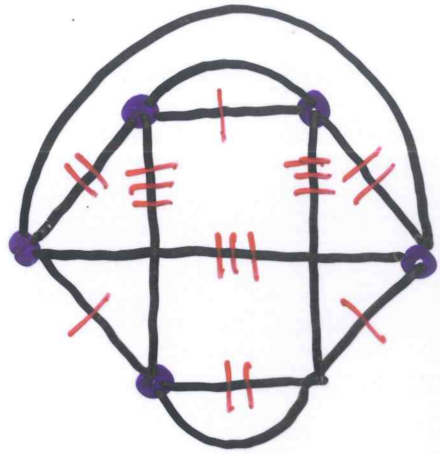
THEOREM (C.-M.)

ANY COMPACT (POSSIBLY CLOSED) 3-MANIFOLD WITH NON-SPHERICAL BOUNDARY COMPONENTS ADMITS REPRESENTATION VIA 4-COLORED GRAPHS

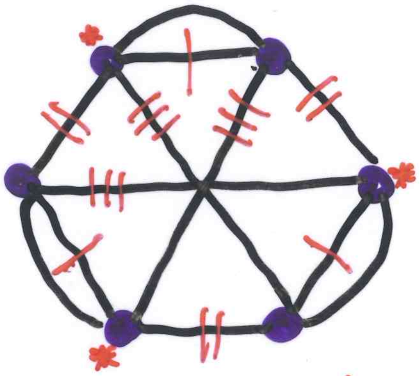
EXAMPLES



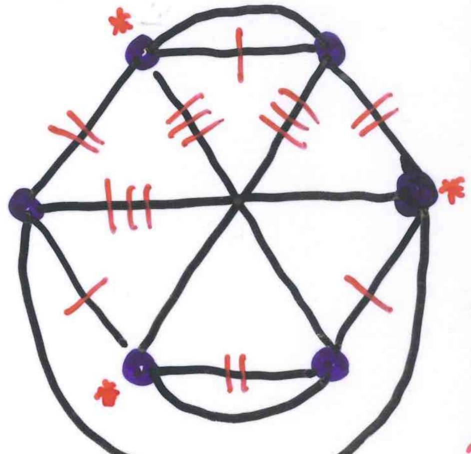
$\mathbb{RP}^2 \times I$



$\widetilde{H}_1$



$S^1 \times S^1 \times I$

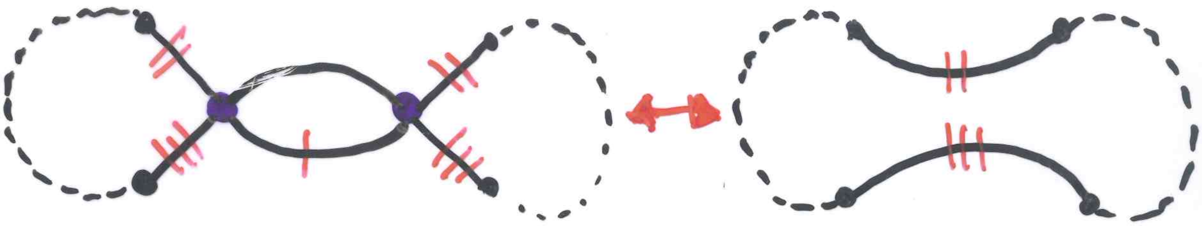


H_1
$\cong$
$D^2 \times S^1$

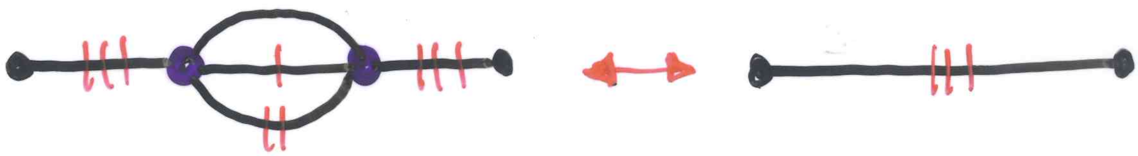
THEOREM (C.-M.)

M_Γ is orientable $\Leftrightarrow$ Γ is bipartite

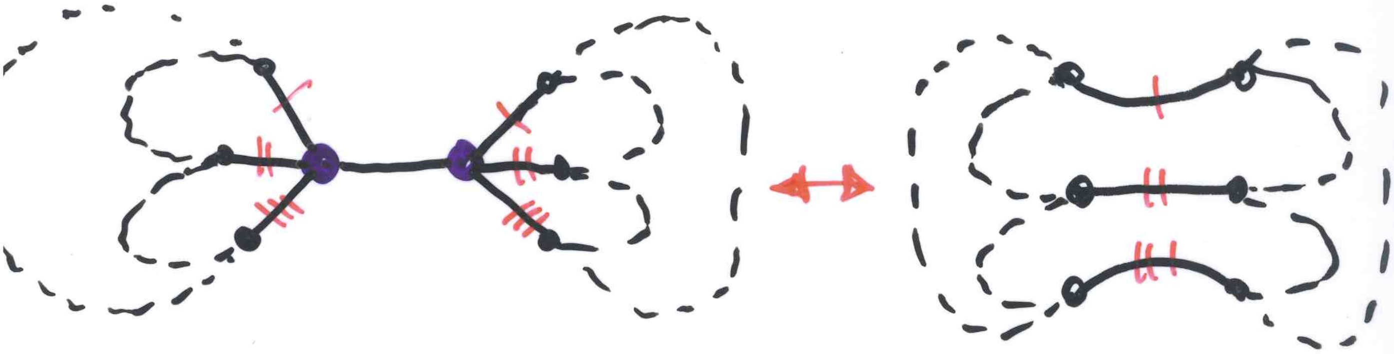
MOVES



II



III



I

Moves are PROPER (or ADMISSIBLE) if the manifold does not change after the application of the move.

THEOREM (C.-M.)

Moves of type II and III are always proper.
Moves of type I is proper if and only if at least one of the two 3-residues involved is ordinary ($\cong S^2$)

REMARK: i) In the closed case moves are sufficient to relate two graphs representing the same manifold. (CASALI '97)

ii) In the non-closed case the property is no longer true because moves don't change the colors of the involved 3-residues (which correspond to boundary components for singular 3-residues)

COMPUTATIONAL RESULTS $(\partial M \neq \emptyset)$

	2	4	6	8	10	12
BIP.	0	0	2	4	57	903
NON BIP.	0	1	6	90	3967	395881

NON ISOMORPHIC GRAPHS

THEOREM (C.-M.)

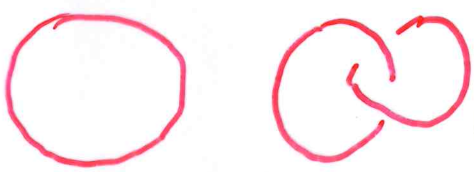
i) There are exactly 7 non-orientable compact 3-manifolds, up to six vertices of the graph representation.

ii) There are exactly 5 orientable compact 3-manifolds, up to eight vertices of the graph representation.

THEOREM (C.-F.-M.-T.)

The 5 manifolds of ii) are the complements of the following links in S^3:

6 VER.



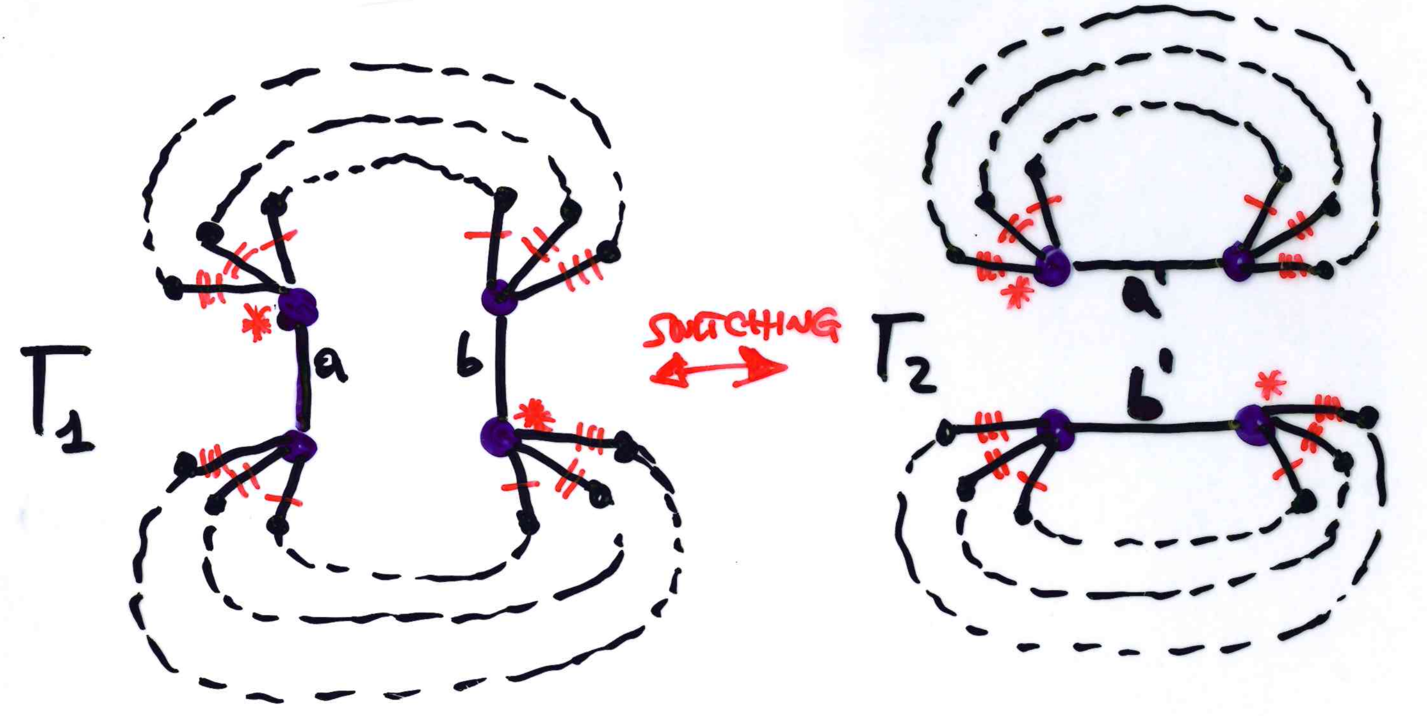
H

8 VER.

H#H L8n8 L8n7

ρ_3-PAIR SWITCHINGS

(ORIENTABLE/BIPARTITE CASE)



The edges a, b of the ρ_3-pair belong to the same three 3-residues.

Let r be the number of the ordinary ones (index) $(0 \le r \le 3)$

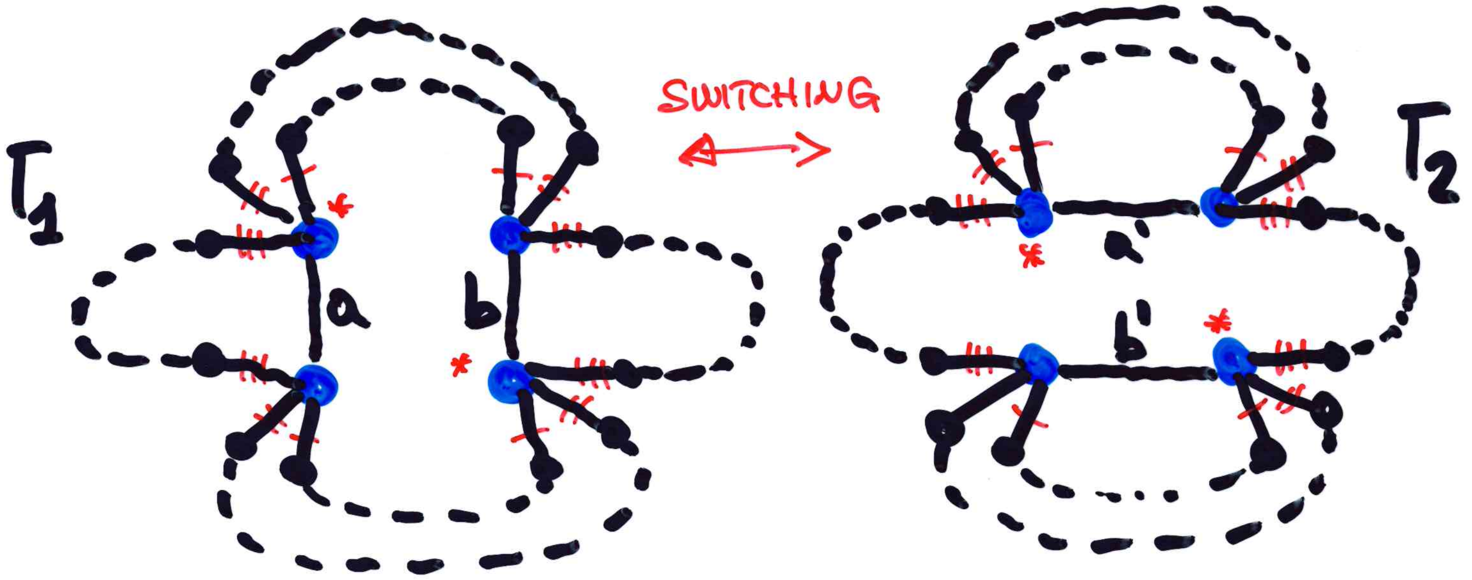
THEOREM (C.-F.-M.-T.)

A) Let $r=3$, then:
- i) If Γ_2 is disconnected $(\Gamma_2 = \Gamma_2' \cup \Gamma_2'')$ then $M_{\Gamma_1} \cong M_{\Gamma_2'} \# M_{\Gamma_2''}$
- ii) If Γ_2 is not disconnected then $M_{\Gamma_1} \cong M_{\Gamma_2} \# (S^2 \times S^1)$

B) Let $r=2$ and the singular 3-residues correspond to a torus.
then
- i) If Γ_2 is disconnected $(\Gamma_2 = \Gamma_2' \cup \Gamma_2'')$ then $M_{\Gamma_1} \cong M_{\Gamma_2'} \# M_{\Gamma_2''}$
- ii) If Γ_2 is not disconnected then either $M_{\Gamma_1} \cong M_{\Gamma_2} \# (S^2 \times S^1)$ or $M_{\Gamma_1} \cong M_{\Gamma_2} \# (D^2 \times S^1)$

ρ_2-PAIR SWITCHINGS

(ORIENTABLE/BIPARTITE CASE)



A ρ_2-pair is called **GOOD** if the associated 3-residue splits into two connected components and at least one of them is ordinary ($\cong S^2$)

A ρ_3-pair in a 4-colored graph representing a 3-manifold with toric boundary is called **GOOD** if its index is $r \geq 2$.

A 4-colored graph is called **RIGID** if it contains no good ρ_2- and ρ_3-pairs.

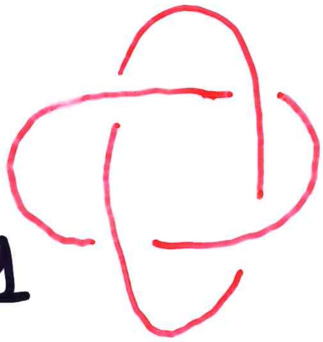
THEOREM (C.-F.-M.-T.)

Any minimal 4-colored graph of a compact orientable prime and boundary-prime 3-manifold $\neq S^2 \times S^1, D^2 \times S^1$ is rigid

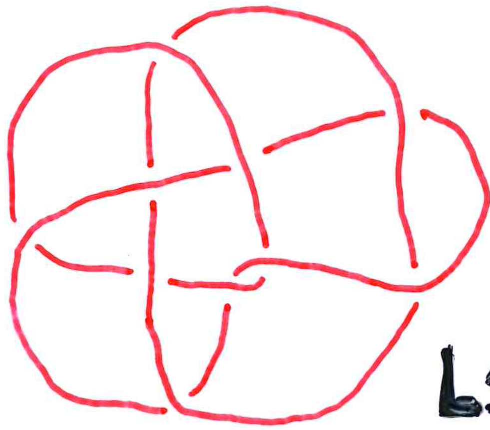
ORIENTABLE CASES - TORUS BOUNDARY

10 VERTICES

20 GRAPHS → 3 MANIFOLDS → 2 PRIME AND NEW
(8 RIGID)



L4a1



L10n93

12 VERTICES

i) CONNECTED BOUNDARY

26 GRAPHS → 3 MANIFOLDS → 1 PRIME AND NEW
(1 RIGID)

$S(D^2, (2,1), (2,1))$

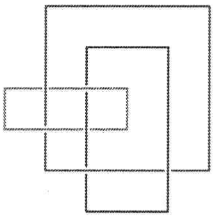
IT IS NOT A COMPLEMENT OF A LINK IN S^3

ii) NON CONNECTED BOUNDARY

148 GRAPHS → 27 MANIFOLDS → 17 PRIME AND NEW
(92 RIGID)

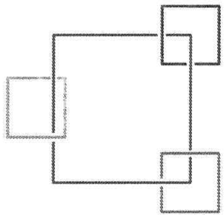
15 COMPLEMENTS OF LINKS IN S^3 →

11

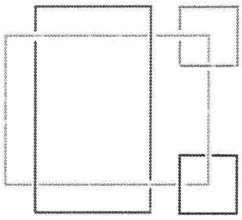


L6a5

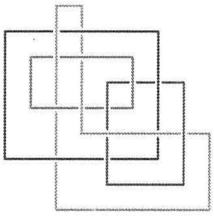
12^3_1



12^4_1

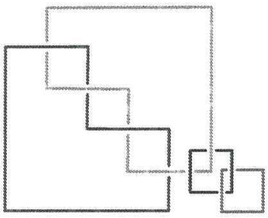


12^4_2

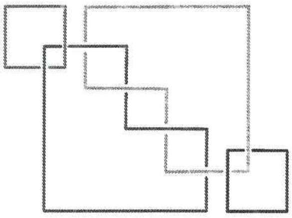


L14n62853

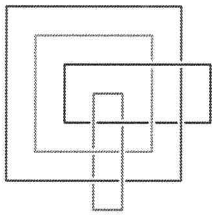
12^4_3



12^4_4

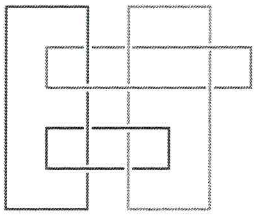


12^4_5



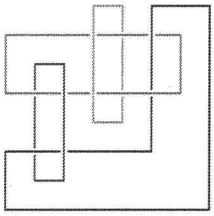
L10n111

12^4_6



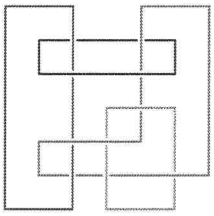
L10n98

12^4_7



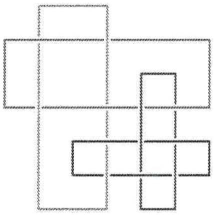
L10n100

12^4_8



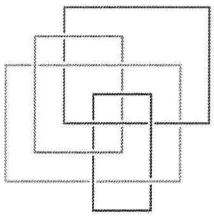
L10n101

12^4_9



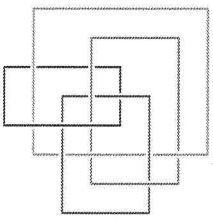
L12n2201

12^4_{10}



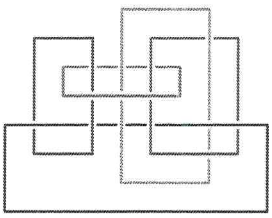
L12n2205

12^4_{11}



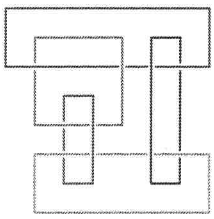
L12n2208

12^4_{12}



L14n63756

12^5_1



L10n113

12^5_2

FIGURE 6. links.

ORIENTABLE CASE - CONNECTED TORUS BOUNDARY

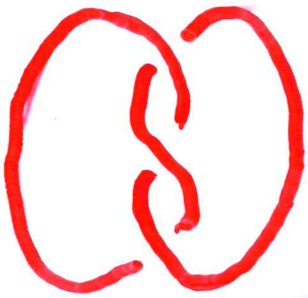
14 VERTICES

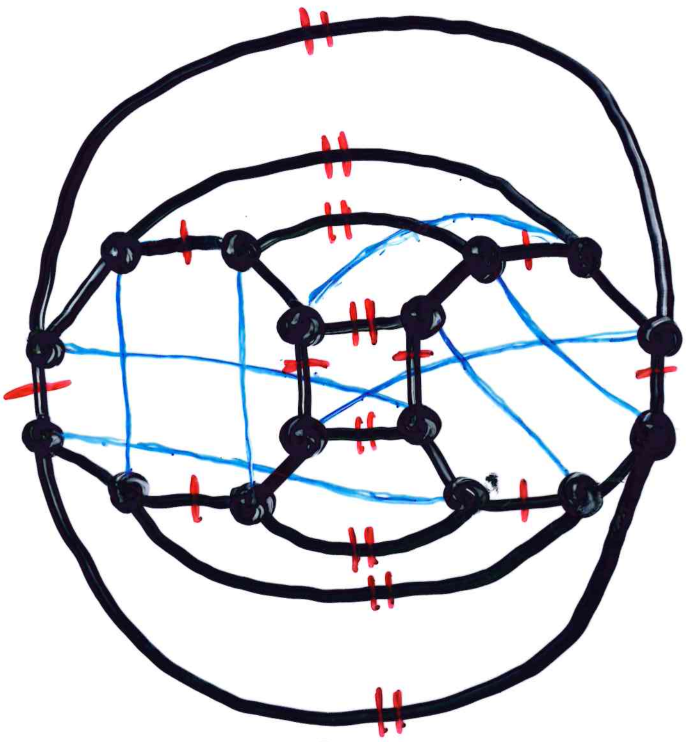
13 GRAPHS → NO PRIME NEW MANIFOLD OCCURS!
(0 RIGID)

16 VERTICES

84 GRAPHS → 1 PRIME NEW MANIFOLD:
(2 RIGID)

THE COMPLEMENT OF THE TREFOIL KNOT





GRAPH OF $S^3 - 3_1$

2p	2	4	6	8	10	12
$C^{(2p)}$	0	0	2	4	57	902
$\tilde{C}^{(2p)}$	0	1	6	90	3967	395877

2p	6	8	10	12	14	16
$C_t^{(2p)}$	2	4	20	174	1979	24058
$C_{tc}^{(2p)}$	1	0	0	26	13	84
$C_{rt}^{(2p)}$	1	4	8	93	1391	4695
$C_{rtc}^{(2p)}$	0	0	0	1	0	2

Name	Code	Manifold	Link
6_1^1	CABCBABCA	$D^2 \times S^1$	unknot
6_1^2	CABCABBCA	$D_1^2 \times S^1$	L2a1
8_1^3	DABCDCABCADB	$D_2^2 \times S^1$	L6n1
8_1^4	DABCDCABCBDA	$(D_2^2 \times S^1) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_2^2 \times S^1)$	L8n8
8_2^4	DABCCDABBCDA	$\texttt{t12047}$	L8n7
10_1^2	EABCDDCEABCDEBA	$\big(D_1^2, (2,1)\big)$	L4a1
10_1^3	EABCDECDABCDEBA	$\big(D_1^2, (2,1)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_2^2 \times S^1)$	L10n93
12_1^1	CABFDEFEDCBAEFABCD	$\big(D^2, (2,1), (2,1)\big)$	–
12_1^2	DABCFEFAECDBBEDFAC	$\big(M_1^2, (1,0)\big)$	–
12_1^3	FABCDEEDFBACDFEACB	$\texttt{s776}$	L6a5
12_1^4	EABCDFFBEADCEFCABD	$D_3^2 \times S^1$	see fig. ??
12_2^4	EABCDFFDEACBBEADFC	$\big(D_1^2, (2,1)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_3^2 \times S^1)$	see fig. ??
12_3^4	EABCDFFDAEBCDCEFBA	$(D_2^2 \times S^1) \bigcup_{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}} (D_2^2 \times S^1)$	L14n62853
12_4^4	EABCDFFEDABCCDEFAB	$\big(D_1^2, (2,1)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_2^2 \times S^1) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_2^2 \times S^1)$	see fig. ??
12_5^4	FABCDEFDAEBCDBEFCA	$(D_2^2 \times S^1) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \big(D_1^2, (2,1)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_2^2 \times S^1)$	see fig. ??
12_6^4	EABCDFFDAEBCCFEBAD	$(D_2^2 \times S^1) \bigcup \texttt{s776}$	L10n111
12_7^4	DABCFEFEABDCEFDACB	$\texttt{o9\_44206}$	L10n98
12_8^4	DABCFEFDEBACCEAFDB	hyperbolic manifold with Vol = 10.991587130	L10n100
12_9^4	DABCFEFEABDCCDEFAB	$otet10_{0014}$	L10n101
12_{10}^4	FABCDEDEFABCCDEFAB	$otet10_{0028}$	L12n2201
12_{11}^4	DABCFEFDEBACECFADB	hyperbolic manifold with Vol = 10.6669791338	L12n2205
12_{12}^4	CABFDEFCEABDDEACFB	$otet12_{0009}$	L12n2208
12_1^5	EABCDFFEABDCCDFEBA	$(D_2^2 \times S^1) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_2^2 \times S^1) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D_2^2 \times S^1)$	L14n63765
12_2^5	DABCFEFEDABCBCFEDA	$otet10_{0027}$	L10n113

Table 3.