Triply even codes obtained from some graphs and finite geometries

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Strongly regular graphs

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- two adjacent vertices have λ common neighbors,
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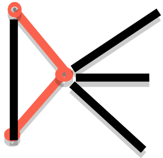
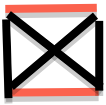
$L(K_6) = T(6) = J(6,2) \quad \binom{6}{2} = 15$
$\mathrm{SRG}(v = 15, k = 8, \lambda = 4, \mu = 4)$





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Let A be the adjacency matrix, and let $C \subset \mathbb{F}_2^{15}$ be the span of the row vectors of A over $\mathbb{F}_2$.
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In general, for $T(n)$,

- Tonchev (1988), Brouwer-van Eijl (1992): $\dim C$,
- Haemers, Peeters and van Rijckevorsel (1999): weight distribution

Even, doubly even, and triply even

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We say that a vector $\boldsymbol{x} \in \mathbb{F}_2^n$ is

$$\begin{aligned} \text{even} &\iff \mathrm{wt}(\boldsymbol{x}) \equiv 0 \pmod 2 \\ \text{doubly even} &\iff \mathrm{wt}(\boldsymbol{x}) \equiv 0 \pmod 4 \\ \text{triply even} &\iff \mathrm{wt}(\boldsymbol{x}) \equiv 0 \pmod 8 \end{aligned}$$

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(denoting by $*$ the entrywise product)

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This is because the mapping

$$f : \{\text{even vectors in } \mathbb{F}_2^n\} \to \mathbb{F}_2$$

defined by

$$f : \boldsymbol{x} \mapsto \frac{\mathrm{wt}(\boldsymbol{x})}{2} \bmod 2$$

is a quadratic form.

$$f(\sum_{i=1}^{k} a_i r_i) = \sum_{i=1}^{k} a_i^2 f(r_i) + \sum_{i<j} a_i a_j \,\mathrm{wt}(r_i * r_j).$$

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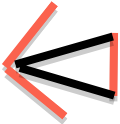
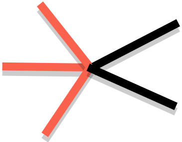
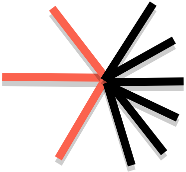


$T(4n+2)$





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Sharp contrast with doubly even codes: rate is always $\approx 1/2$.
There are triply even codes with rate $1/4$ whenenver $n \equiv 0 \pmod{16}$.

Classification of triply even codes of length 48

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There are 10 *maximal triply even codes of length* 48.

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Motivation comes from Framed Vertex Operator Algebras (FVOA).

- The moonshine module $V^\natural$ has Virasoro frames, and each Virasoro frame gives rise to a triply even code of length 48.
- Lam–Yamauchi (2008) showed that, conversely, every triply even code of length divisible by 16 is obtained from some FVOA.
- The classification lead Lam and Shimakura to discover new FVOA≈CFT conjectured by Schellekens (1993).

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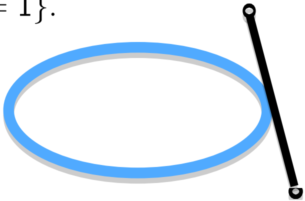

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$$\begin{aligned} &\{\pm x\} \sim \{\pm y\} \\ &\iff Q(x \pm y) = 0 \\ &\iff \text{the line through } \langle x \rangle \text{ and } \langle y \rangle \text{ is a "tangent"} \\ &\iff \text{the line } \langle x, y \rangle \text{ and the surface } Q = 0 \text{ in } PG(3,3) \\ &\qquad \text{has exactly one point in common} \end{aligned}$$

$PGO_4^-(q)$, q an odd prime power

Let $V = \mathbb{F}_q^4$ be equipped with a nondegenerate quadratic form with Witt index 1, for example, with $\eta \notin (\mathbb{F}_q^\times)^2$,

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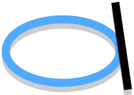

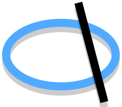

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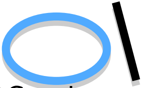

Then $|X| = q(q^2 + 1)/2$. Adjacency by tangent. Not SRG unless $q = 3$. Brouwer–Cohen–Neumaier, Section 12.2 shows this is a 3-class association scheme.

Theorem (Betsumiya–M.)

For any odd prime power q, the code of this graph is triply even, of dimension at least $(q^2 - 1)/2$.

Proof: k, λ, μ (BCN, Section 12.2)

Let $V = \mathbb{F}_q^4$ be equipped with a nondegenerate quadratic form Q with Witt index 1. Define a graph Γ whose vertex set is

$$X = \{\{\pm x\} \mid Q(x) = 1\},$$

with adjacency

$$\{\pm x\} \sim \{\pm y\} \iff B(x, y) = \pm 1,$$

where

$$B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)).$$

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The graph has valency $q^2 - 1 \equiv 0 \pmod 8$, $\lambda = 2(q - 1) \equiv 0 \pmod 4$, $\mu = 2(q - 1)$ or $2(q + 1) \equiv 0 \pmod 4$, depending on $\langle x, y \rangle$ is external or secant.

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Proof: the number of common neighbors of three vertices is even

Let $\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle$ be distinct vertices, Their common neighbors are

$$\{\langle z \rangle \mid Q(z) = 1,\ B(x_i, z) = \pm 1 \ (i = 1, 2, 3)\}.$$

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$$\{\langle x_0 + y \rangle \mid Q(y) = 1 - Q(x_0),\ y \in W^\perp\} : \text{ even}$$

Proof: the number of common neighbors of three vertices is even

Let $\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle$ be distinct vertices, Their common neighbors are

$$\{\langle z\rangle \mid Q(z) = 1,\ B(x_i, z) = \pm 1\ (i = 1, 2, 3)\}.$$

For simplicity, assume $W = \langle x_1, x_2, x_3\rangle$ is a nondegenerate 3-dimensional subspace, and consider

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$V = \mathbb{F}_q^4$ is equipped with a nondegenerate quadratic form Q with Witt index 1. The vertex set of Γ is $X = \{\{\pm x\} \mid Q(x) = 1\}$, with adjacency

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$$\begin{aligned}Y&=\{\langle\lambda x+y\rangle\mid\lambda\in\mathbb{F}_q^\times,\ y\in V_-,\ Q(y)=1\}\\&=\{\langle\lambda x+y_i\rangle\mid\lambda\in\mathbb{F}_q^\times,\ 1\leq i\leq(q+1)/2\},\end{aligned}$$

since

$$B(\lambda x+y_i,\mu x+y_j)=B(y_i,y_j)=\begin{cases}1 & \text{if } i=j,\\ \text{not }\ \pm 1 & \text{otherwise.}\end{cases}$$

Maximality?

Since $\dim C \geq (q^2-1)/2$, the rate is at least

$$\frac{\frac{q^2-1}{2}}{\frac{q(q^2+1)}{2}} = \frac{q^2-1}{q(q^2+1)} \to 0 \quad (q \to \infty).$$

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Thank you for your attention!