IS THERE A (4,27,2) PARTIAL GEOMETRY?

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A Hierarchy of Problems

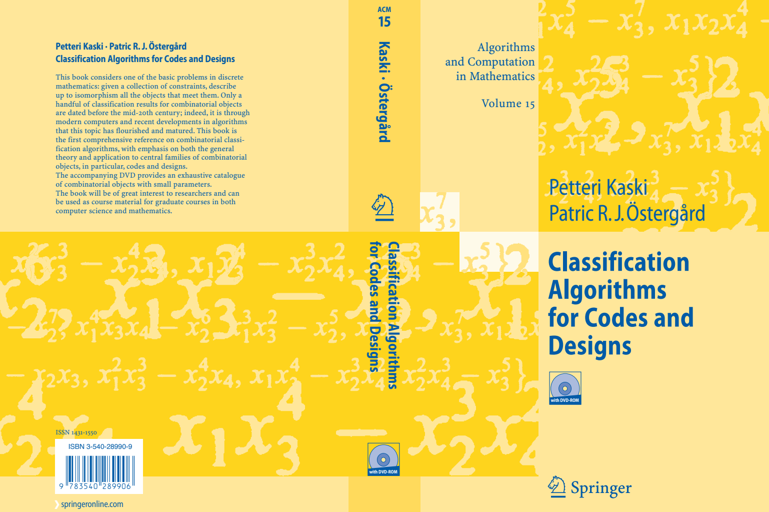
Existence problem Given a collection of properties, decide whether there exists an object satisfying these properties.

Counting problem Given a collection of properties, count the number of distinct objects satisfying the properties. Two versions: all, all up to isomorphism/equivalence.

Classification problem Given a collection of properties, describe, up to some criterion of isomorphism, all the objects that have the desired properties.

Characterization problem Develop a deeper understanding of classified objects.

Classification



Partial Geometries

An (s, t, α) partial geometry, briefly $\mathrm{pg}(s, t, \alpha)$, is a point-line incidence structure such that

1. there are exactly $s + 1$ points on each line;
2. there are exactly $t + 1$ lines through each point;
3. every pair of distinct points are on at most one line; and
4. for every line L and every point p not on L, there are exactly α lines through p meeting L.

There is a point-line duality for partial geometries. The number of points v and lines b follow from the main parameters of the partial geometry.

Some Known Results

v	s	t	α	b	Existence
15	2	2	1	15	Unique
27	2	4	1	45	Unique
28	3	4	2	35	Does not exist
40	3	3	1	40	Exactly two exist
45	4	6	3	63	Exactly two exist
64	3	5	1	96	Unique
66	5	8	4	99	Does not exist
70	6	6	4	70	?
75	4	7	2	120	Does not exist
76	3	6	1	133	Does not exist
			$\vdots$		
275	4	27	2	1540	?
			$\vdots$		

McLaughlin Structures

McLaughlin group A simple group of order 898 128 000.

McLaughlin graph The unique strongly regular graph Γ with parameters $v = 275$, $k = 112$, $\lambda = 30$, $\mu = 56$, briefly a $(275, 112, 30, 56)$ srg.

McLaughlin geometry A $\mathrm{pg}(4, 27, 2)$; has the McLaughlin graph as its point graph.

Open Problem

Is there a McLaughlin geometry?

History

1968 Jack McLaughlin [A simple group of order 898,128,000. 1969 Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968) pp. 109–111, Benjamin, New York]

1984 Jack van Lint

1997 Rudolf Mathon

1997 Sven Reichart

2006 Leonard Soicher

2009 Christian Pech and Sven Reichart

2012 Patric Ö. and Leonard Soicher (1. attempt)

2015 Patric Ö. and Leonard Soicher (2. attempt)

Some Facts

Fact

A McLaughlin geometry has 275 points and 1540 lines.

Fact

The candidates for the lines are the 5-cliques of Γ. There are 15400 such 5-cliques.

Fact

To construct a McLaughlin geometry, one needs to find a partition of the 15400 edges of Γ into 1540 of the 15400 5-cliques.

Note that the order of the symmetry group that we can utilize is $1.8 \cdot 10^9 \approx |\mathrm{Aut}(\Gamma)| \ll |S_{275}| \approx 1.0 \cdot 10^{553}$.

Exhaustive Search for a McLaughlin Geometry (1)

There are 17 729 280 sets of 28 candidate lines (called *bundles*) through any point p of the geometry ($\mathrm{Aut}(\Gamma)$ acts transitively on the vertices of Γ). These bundles are partitioned into 36 U-orbits, where U is the stabilizer in $\mathrm{Aut}(\Gamma)$ of the point p.

$$
\begin{array}{l}
111111111111111111111111111100000\cdots\\
100000000000000000000000000000000\\
100000000000000000000000000000000\\
100000000000000000000000000000000\\
100000000000000000000000000000000\\
010000000000000000000000000000000\\
010000000000000000000000000000000\\
010000000000000000000000000000000\\
010000000000000000000000000000000\\
\vdots
\end{array}
$$

Exhaustive Search for a McLaughlin Geometry (2)

The 36 bundles obtained, up to symmetry, after completing one point do not have much symmetry left:

$\lvert Stab \rvert$	#	$\lvert Stab \rvert$	#	$\lvert Stab \rvert$	#
2	1	16	6	48	4
3	1	18	4	64	1
6	3	21	2	192	2
7	1	24	2	432	2
14	4	32	2	12 096	1

Exhaustive Search for a McLaughlin Geometry (3)

Algorithm: Complete one point at time.

Which points to complete after the first point?

- Consider points p' such that $\{p, p'\}$ is not an edge in Γ for any points p completed earlier. (The largest independent set in Γ has size 22, and the search will never get that far.)
- Carry out (partial) isomorph rejection after the second point has been completed, but not later.
- Find 9 additional points to complete.

Exhaustive Search for a McLaughlin Geometry (4)

Note 1. Starting from scratch, we are essentially asking whether a given 11-partite graph of order $11 \cdot 17\,729\,280 = 195\,022\,080$ has a clique of size 11.

Note 2. Starting from the (about 10^6) structures with two completed points, we are essentially asking whether a given 9-partite graph of order about 500 000 has a clique of size 9.

Exhaustive Search for a McLaughlin Geometry (4)

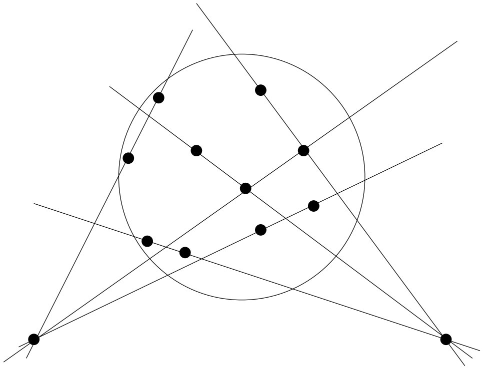
Some details and techniques:

- The 36 bundle types are labeled from 1 to 36 (using some heuristic), and the search is divided into 36 parts. In part i, no line sets with label smaller than i are considered.
- Various auxiliary data structures are essential in tuning the approach.
- Maintaining the edges of graphs explicitly would consume too much time and memory (cf. previous Note 2).

Compatibility of Bundles (1)

Fact

Let P be the set of joint neighbors of two nonadjacent vertices in Γ. The subgraph of Γ induced by P is an $(56, 10, 0, 2)$ graph, the (unique) Gewirtz graph.



Compatibility of Bundles (2)

Precomputing compatibility of bundles: too time- and memory-consuming.
Precomputing nothing (exhaustive comparison on-the-fly): too time-consuming.

Solution: Partial precomputing. For each pair of points/vertices, p and q, and each bundle through one of them, form a 280-bit string with 1s telling which 28 out of the 280 edges of the Gewirtz graph are covered by the bundle.

Adjacency test in C:

```
if(((*p1)&(*p2))||((*(p1+1))&(*(p2+1)))||
((*(p1+2))&(*(p2+2)))||((*(p1+3))&(*(p2+3)))||
((*(p1+4))&(*(p2+4))))
```

Is There a McLaughlin Geometry?

- A 256-core cluster with 2.4-GHz Intel Xeon E5-2665 processors was used for the search.
- The search started in February 2015.
- The search ended in March 2016.

Result

There is no McLaughlin geometry.

One Impact of the Result

The point graph of a pg(s, t, α) is a strongly regular graph with parameters

$$((s+1)(st+\alpha)/\alpha,\ s(t+1),\ s-1+t(\alpha-1),\ (t+1)\alpha)\,. \qquad (1)$$

pseudo-geometric graph A strongly regular graph with parameters (1).

geometric graph A pseudo-geometric graph that is the point graph of at least one pg(s, t, α).

Krein bound The parameters of a pseudo-geometric graph satisfy $(s+1-2\alpha)t \leq (s-1)(s+1-\alpha)^2$.

If $\alpha = 1$—giving a *generalized quadrangle*—then a pseudo-geometric graph attaining the Krein bound is geometric. Does this hold for $\alpha > 1$ as well (asked by Cameron–Goethals–Seidel)?

NO.

Thank You!

Спасибо!

First attempt: Близо́к локото́к, да не уку́сишь.
Second attempt: Без труда́ не вы́тащишь и ры́бку из пруда́.

Further Tricks