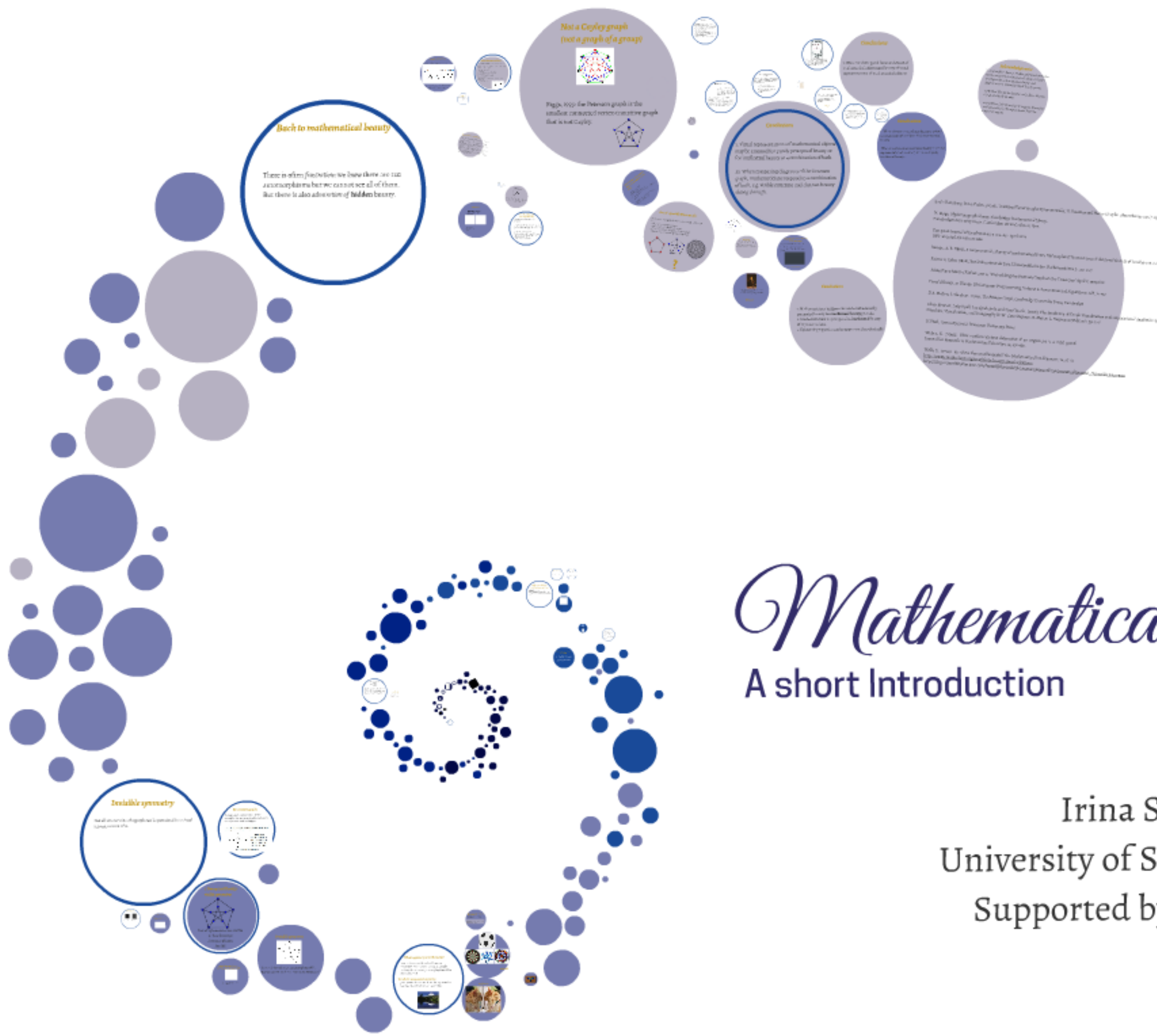
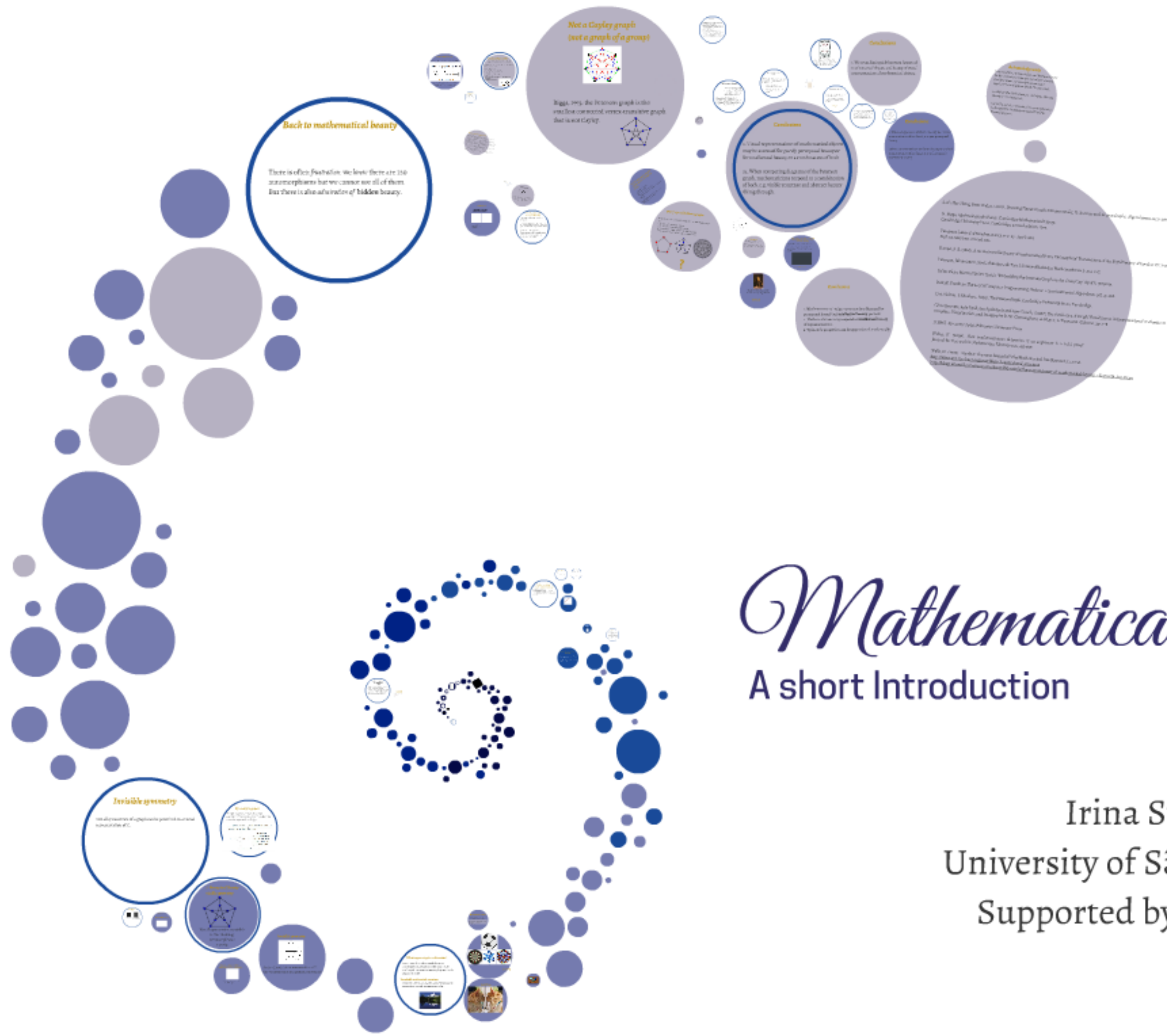


Mathematical Beauty

A short Introduction

Irina Starikova
University of São Paulo
Supported by CAPES



Mathematical Beauty

A short Introduction

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Old and difficult but interesting problem.

Mathematicians about mathematical beauty

G.H.Hardy, *A Mathematician's Apology* (1993)

"The mathematician's patterns, like the painter's or the poet's must be *beautiful*; the ideas, like the colors or the words must fit together in a harmonious way."

Terrence Tao (2007, p. 624): "What Is Good Mathematics?"

21! features including *elegance* and *beauty*.

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er of taste.

Observations

1. Mathematicians insist on genuine beauty in mathematics and its **importance**:

Mathematical beauty is a "real aesthetic feeling that all true mathematicians recognize" (Henri Poincare in *Science and Method* 1914, p. 59).

2. Mathematicians consider mathematical beauty as a **guide** in their research:

"Beauty is the *first test*: there is no permanent place in this world for ugly mathematics" (Hardy, p.14).

"You can recognize truth by its beauty and simplicity." (Richard Feynman)

Conditions of Beauty

A *power* or a combination of *qualities*: shape, color, form, that *pleases* the *aesthetic senses* (sight, hearing, intellect or moral sense).

Attributes

a) Beautiful object

b) Sensitive subject

c) Aesthetic pleasure

Non-attributes

Dis-interested (not an appetite or a gain).

It is not just a matter of taste.

Pleasure of doing mathematics

“The pleasure of doing mathematics” was rated by

91% as “absolutely important” or "very important"
62% "a very pleasurable sort of game"
32% "an inner necessity" (Bernard Zarca 2011).

What kind of pleasure is this?

Specifics of mathematical aesthetics and related skepticism

1. Abstract/**non-perceptual;**
2. Rule-following activity;
3. Epistemic **aims**, rationality, truth are essential;
4. Limited to **experts'** judgement (makes the discussion more difficult).

Skepticism

1. Sensory dependance argument (physicalism)

There must be some physical/*sensory* properties of the object that entails beauty: strokes of paint, shades, coloures, sounds (Zangwill "Aesthetic/ Sensory Dependence" 1998). Maths cannot be beautiful because it is abtstract!

2. Epistemic function argument

"[W]hat we are appreciating in [an elegant proof] ... is a pleasure of finding a very *effective means* to an end." Zangwill, The Metaphysics of Beauty, p. 142.

The aim of this talk is to reply a skeptic.

A reply to the first objection:
Concerning aesthetic sensitivity

We should not restrict the domain of "beauty" because experts may see more beyond perceptual beauty:

"To those who do not know mathematics it is difficult to get across a real feeling... to *the deepest beauty*..." (The Feynman Lectures on Physics (1964) Ch2. "The Relation of Mathematics to Physics," p. 58)

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A reply concerning elegant proofs

1. Formally: a failed proof is not a "proof".
2. Emotionally: the A. judgement looses its strength.
3. Objectively: the beauty does not disapper; it may reencarnate in a modification of the failed argument.

A reply to the second objection

- Philosophers concentrate on *proofs*. However mathematicians talk about beauty not only in respect to proofs, but also about:
- **statements** of theorems, conjectures
- **structures**/objects
- **ways** of thinking

Beautiful proofs=beautiful structures+beautiful way of thinking?

Yuri Manin *Mathematics as a Metaphor* 2008

Serge Lange *The beauty of doing Mathematics* 1985

Adrian Bondy *Beautiful conjectures in graph theory* 2014

Michael Harris *Mathematics without Apologies* 2015

Strategy:

1) Consider an object (not a proof), which is beautiful independently of its epistemic purpose (e.g. a graph).
2) Show that it is beautiful not for perceptual properties but for its abstract properties.

Mathematical Objects and their Representations

Mathematical **object** - abstract,
Visual **representation** - physical and cognitive: drawings, models, geometric, graphical or symbolic elements.

Perceptual vs. intellectual beauty

Perceptual beauty can be found in visually pleasing representations, formulas.

Intellectual beauty is about ***abstract*** mathematical objects. ❤

Approach

i) Take a "beautiful" graph (experts' opinion).

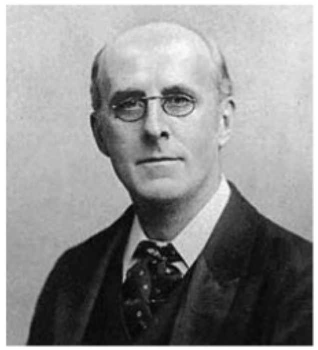
ii) Consider a property which has traditionally been a factor of beauty – e.g. **symmetry**.

iii) Distinguish between **perceived** and **calculated** (*geometrical* and *combinatorial*) symmetry.

iv) See how *geometrical* and *combinatorial* symmetries contribute to the beauty.

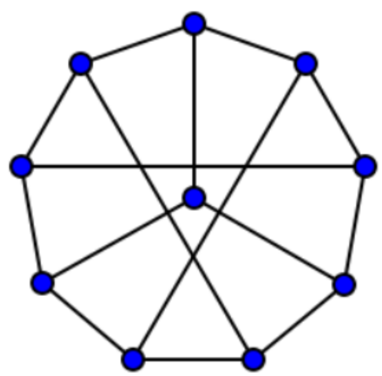
what graph?

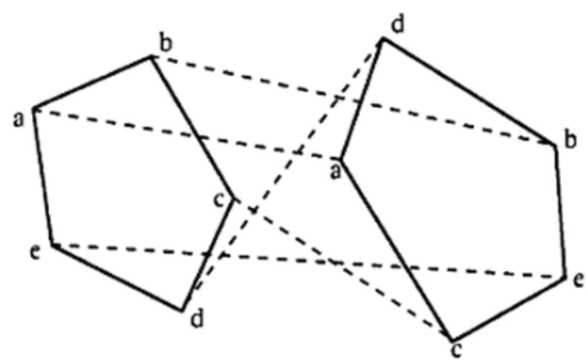
A case study: the Petersen graph



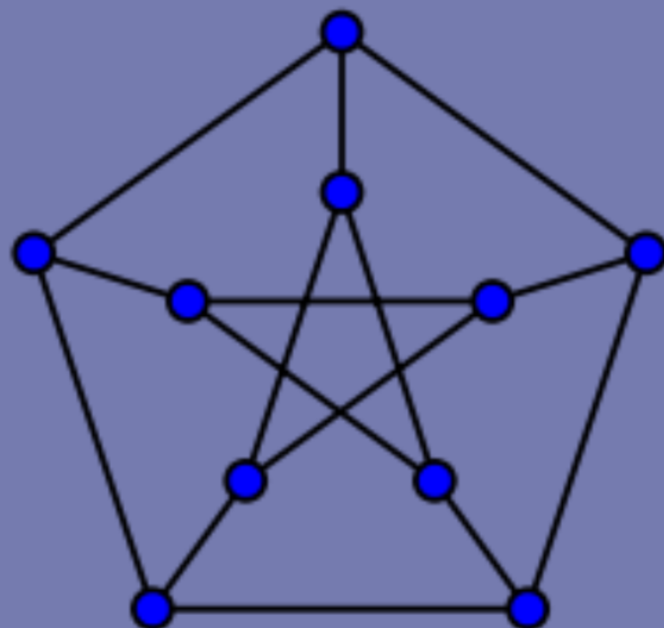
Kempe, A. B. (1886)

Petersen, Julius (1898)





1. Perceptual Beauty: visible symmetry



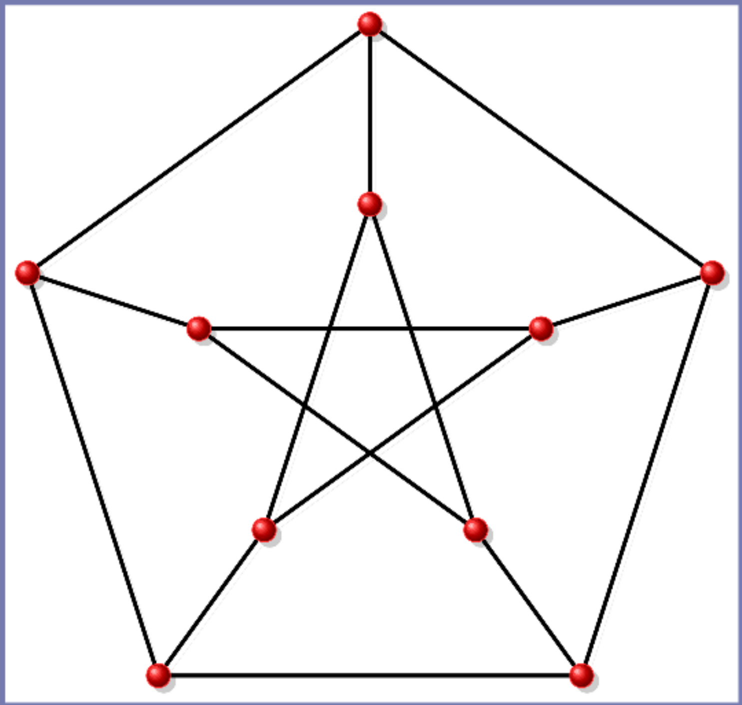
Not all symmetries are visible in this drawing:

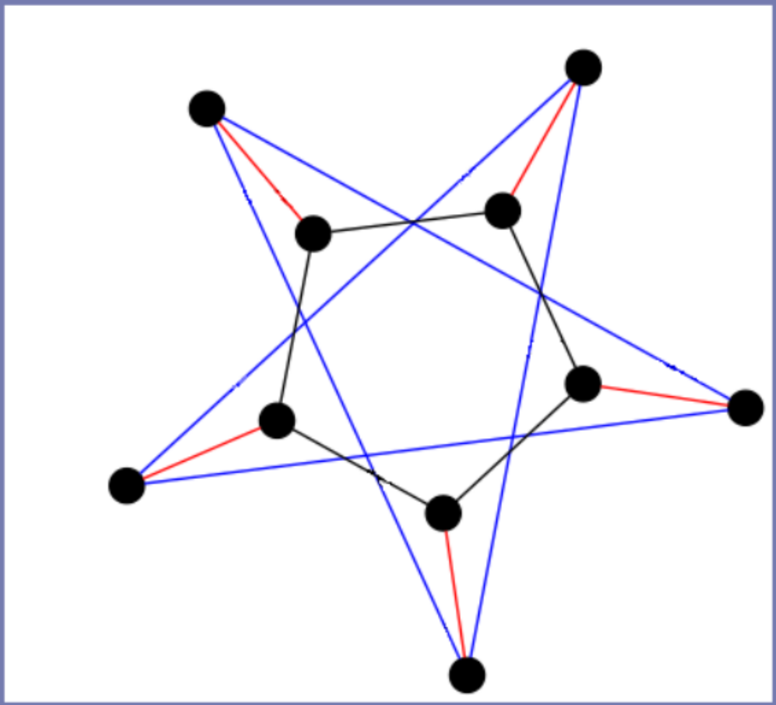
Automorphisms

120 (S5)

The Beautiful and the Ugly

(Despite the same number of symmetries)





- Clear subgraphs
- Less edge intersections
- Broader angles

"What about the invisible symmetries? How can they contribute to beauty?"

They contribute to the beauty not of a visual representation, but of the object represented.

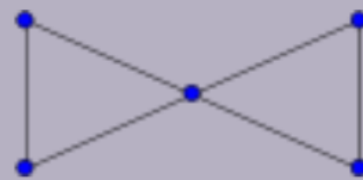
- symmetry helps us to grasp mathematical structure
- symmetry implies other significant properties and connections

What makes this graph beautiful (as an abstract object)?

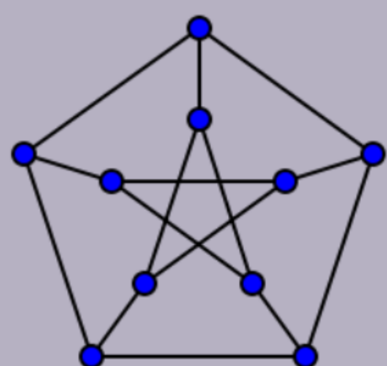
- Rich in symmetry: regular, vertex-, edge-, distance-transitive;
- One of only 13 cubic distance-regular graphs;
- Non-Cayley;
- One of 5 vertex-transitive connected non-Hamiltonian graphs;
- Hypohamiltonian;
- Optimal network graph;
- ...

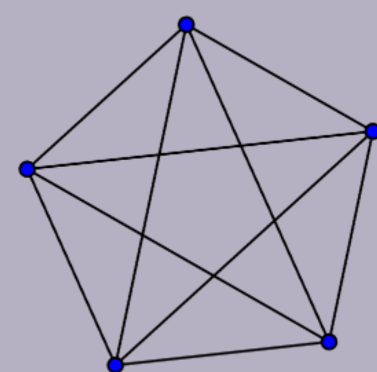
Non-planar

A planar graph can be visualised on a plane without edge crossings.



The Petersen graph is non-planar. Having K5 or K3,3 as a minor proves non-planarity (visible with the aid of symmetry of P).





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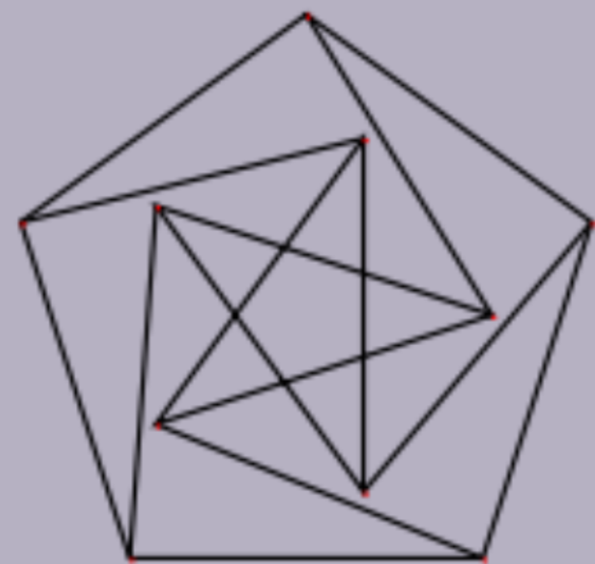
A unit distance graph

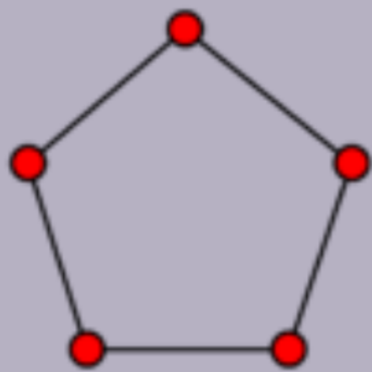


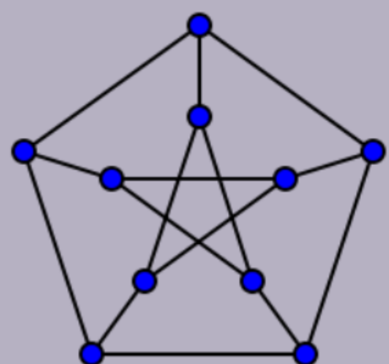
The Petersen graph is a unit distance graph: it can be drawn in the plane with each edge having unit length (from the symmetry of order 5).

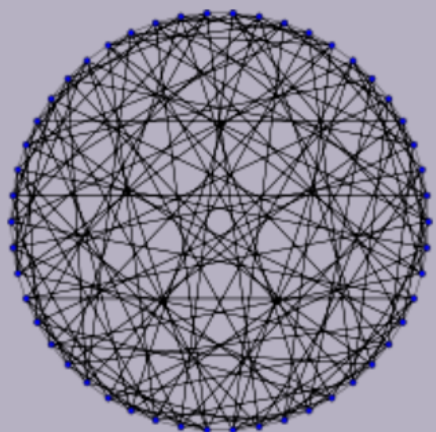
One of 4 possible Moore graphs

For diameter=2 and girth=5 the only possible strongly regular graphs are:

1) The 5-vertex 2-regular pentagon;
2) The 10-vertex 3-regular Petersen graph;
3) The 50-vertex 7-regular Hoffman-Singleton graph;
4) The 3250-vertex 57-regular graph (open problem)

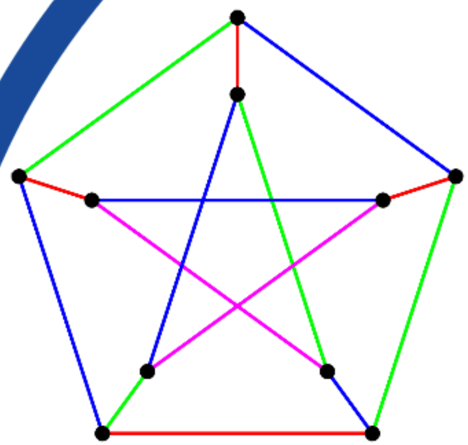






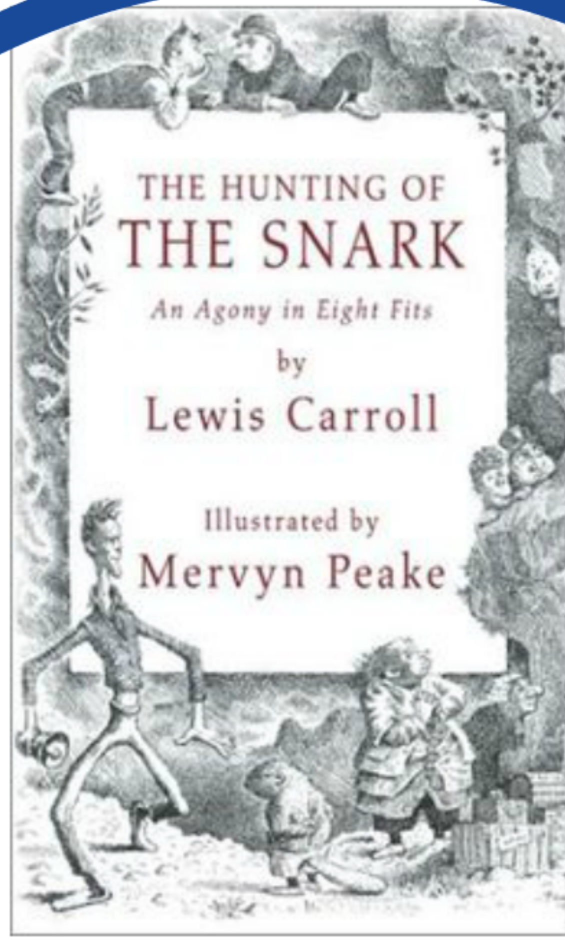
?

The Petersen graph is a snark



A *snark* is a cubic bridgeless graph in which every vertex has 3 neighbors, and the edges cannot be colored by only 3 colors without 2 edges of the same color meeting at a point.

"In the study of various important and difficult problems in graph theory one encounters an interesting but somewhat mysterious variety of graphs called *snarks*. In spite of their simple definition… and over a century long investigation, their properties and structure are largely unknown (Chladny, 20…).



"After crossing the Ocean the crew of 10 arrives in a strange land and learn about the five signs of a Snark: meagre and hollow, but crisp taste; a habit of rising late and taking breakfast during five o'clock tea; ambitious..." Lewis Carroll *The Hunting of Snark*

The Petersen graph is the "main" snark(?)

P. G. Tait initiated the study of snarks in 1880, when he proved that the **four color theorem is equivalent to the statement that *no snark is planar***. The first known snark was the *Petersen* graph, discovered in 1898.

W. T. Tutte conjectured that *every snark has the Petersen graph as a minor.*

(In 1999, Robertson, Sanders, Seymour, and Thomas announced a proof of this conjecture.)

The following are factors of the abstract beauty of the Petersen graph:

- Small but non-trivial
- Highly symmetric
- Many interesting properties
- A counter-example for numerous conjectures
- Unique

Interestingly many of these propetties are visible in the favorit drawing!

A check list

A **beautiful object** which has a power to affect a sensitive subject with **aesthetic pleasure**? **-Yes.**

"The Petersen graph is a **beautiful** graph, ...a graph theorists will tell you, time and again... [I]n **spite of its small size** its structure is **beautifully symmetric**, and this has far reaching consequences" Erickson [2014].

1. Is it intellectual beauty? -Yes.

"If you are talking about **theoretical beauty**, Petersen graph would top the list. Simple, but almost always counterexample to the simple theorems you try to cook up" Ashwin [2014].

2. Loose use? Practical gain? -No!

The properties are there, whether you use the graph ir not.

Concluding replies to a sceptic

1. Mathematical **abstract** objects/structures **can** be beautiful independently of their efficacy.

2. Epistemic properties (simplicity, clarity, understanding) can be **conditions** for aesthetic appreciation.

Thank you!

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Acknowledgments

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intellectual beauty

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intellectual beauty

preciated aesthetically.

II. Introduction to the philosophical discussion.

1. What do mathematicians refer to in their aesthetic-like judgements?
2 Are aesthetic-like judgements **indeed aesthetic** (or epistemic, practical)?

4. Adapt a new perspective on mathematical practice other than "mechanical calculation" but a creative human activity, which engages feelings (*Mathematics as a metaphor* Y. Manin; "Mathematics and Fiction", R.Thomas).