Semigroup property of foliations of vector spaces

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Semigroup property

Let $\mathcal{V}$ be a finite dimensional real vector space and $\mathfrak{F}$ be its foliation. We denote by $\widehat{X}$ the closed convex hull of a set X in $\mathcal{V}$.

Definition

The foliation $\mathfrak{F}$ has the semigroup property if the family $\{\widehat{L} : L \in \mathfrak{F}\}$ is a semigroup, i.e., for any pair of leaves $L_1, L_2 \in \mathfrak{F}$ there exists a leaf $L_3 \in \mathfrak{F}$ such that

$$\widehat{L}_1 + \widehat{L}_2 = \widehat{L}_3.$$

The sum is pointwise: $A + B = \{a + b : a \in A, b \in B\}$.
We abbreviate the semigroup property as SP.

A group $G \subset \mathrm{GL}(\mathcal{V})$ defines the foliation $\mathfrak{F}_G$ of $\mathcal{V}$ whose leaves are the orbits $O_v = Gv$, $v \in \mathcal{V}$. If G is compact, then we may assume that $\mathcal{V}$ is Euclidean and $G \subseteq \mathrm{O}(\mathcal{V})$.

A reflection group is a group generated by orthogonal reflections in hyperplanes (mirrors).

Theorem

Let G be finite. Then $\mathfrak{F}_G$ satisfies SP if and only if G is a reflection group.

SP for foliations defined by group actions

Finite groups

We say that $v \in \mathcal{V}$ is regular if its stable subgroup is trivial.
SP is equivalent to each of the following properties.

Let $\mathrm{cone}(X)$ be the least closed convex cone which contains X and set $T_x X = \mathrm{cone}(\widehat{X} - x)$.

(i) The cone $C_v = T_p \widehat{O}_v$ at the vertex v of $\widehat{O}_v$ is locally constant on $\mathcal{V}^{\mathrm{reg}}$.

(ii) the Dirichlet–Voronoi partition for O_v is independent of $v \in \mathcal{V}^{\mathrm{reg}}$.

The partition consists of the sets

$$D_v = \{x \in \mathcal{V} : |x - v| = \min_{u \in O_v} |x - u|\}.$$

We say that a function f on X has a peak on X at x if $f(x) > f(y)$ for all $y \in X \setminus \{x\}$.

(iii) for any regular v the linear functional $\lambda_v(x) = \langle x, v \rangle$ has a peak on each G-orbit.

Definition

A compact group $G \subset \mathrm{GL}(\mathcal{V})$ is called *polar* if there exists a linear subspace $\mathcal{A} \subseteq \mathcal{V}$ such that

(A) each orbit $O_v = Gv$, $v \in \mathcal{V}$, meets $\mathcal{A}$;

(B) for any $u \in \mathcal{A}$ the tangent space $T_u = T_u O_v$ is orthogonal to $\mathcal{A}$.

Then $\mathcal{A}$ is called a *Cartan subspace*.

Theorem

Let G be connected. Then $\mathfrak{F}_G$ satisfies SP *if and only if G is polar.*

SP for foliations defined by group actions

Compact groups

Let $\mathcal{A}$ be a Cartan subspace for a polar compact group G (this means that its identity component is polar). The Weyl group W is defined as

$$W = \{g \in G : g\mathcal{A} = \mathcal{A}\}\big|_{\mathcal{A}}.$$

Theorem

A compact linear group satisfies SP *if and only if it is polar and its Weyl group is a reflection group.*

SP for foliations defined by group actions

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Let H be a subgroup of $\mathrm{GL}(\mathcal{V})$ and $\mathfrak{C}(\mathcal{V}, H)$ be the family of all H-invariant convex sets in $\mathcal{V}$.

Proposition

If G is polar, then the mapping $Q \to Q \cap \mathcal{A}$, where $\mathcal{A}$ is a Cartan subspace, is a semigroup isomorphism between $\mathfrak{C}(\mathcal{V}, G)$ and $\mathfrak{C}(\mathcal{A}, W)$.

Polar groups and s-representations

Let S be a noncompact semisimple Lie group and G be its maximal compact subgroup. Then $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{v}$, where $\mathfrak{v}$ is $\mathrm{Ad}(G)$-invariant, and $\mathrm{Ad}(G)$ is polar in $\mathfrak{v}$. The Cartan subspaces are the maximal abelian ones.

Definitions

1. The isotropy representation of G in the Riemannian symmetric space S/G is called an *s-representation*.
2. Groups $G, H \subseteq \mathrm{GL}(\mathcal{V})$ are *orbit equivalent* if $\mathfrak{F}_G = \mathfrak{F}_H$.

Theorem (Dadok, 1985)

A representation of a compact semisimple Lie group is polar if and only if it is orbit equivalent to some s-representation.

Polar groups and isoparametric submanifolds

Definition (Terng, 1985)

A submanifold of an Euclidean space is called *isoparametric* if

(1) its normal bundle is flat and

(2) principal curvatures are constant for any parallel normal vector field.

- A principal orbit of a polar representation is isoparametric.
- Conversely, a homogeneous isoparametric submanifold is a principal orbit of a polar action (Palais and Terng, 1987).
- Moreover, any isoparametric submanifold of codimension ≥ 3 is homogeneous (Thorbergsson, 1991).
- There exist non-homogeneous isoparametric submanifolds of codimension 2 (Ozeki and Takeuchi, 1975).

SP for isoparametric foliations

Let $\mathfrak{S}$ be the family of spheres $|x| = r$ in $\mathcal{V}$. We say that a foliation $\mathfrak{F}$ is centered if $\mathfrak{F} \subseteq \mathfrak{S}$.
Any isoparametric submanifold is a leaf of the unique foliation whose principal leaves are isoparametric. If the leaves are compact, then it is subject to a foliation of concentric spheres.

Theorem

A smooth centered foliation with compact leaves is isoparametric if and only if it satisfies SP.

This answers in the affirmative to a question of G. Thorbergsson (formulated in a different but equivalent form).

Centered foliations with finite leaves

Let L_v be the leaf which contains v. We say that a foliation is continuous if the mapping $v \to L_v$ is continuous with respect to the standard topology in $\mathcal{V}$ and the Hausdorff distance between sets in $\mathfrak{F}$.

Theorem

If a foliation $\mathfrak{F}$ with finite leaves is centered and continuous, then $\mathfrak{F} = \mathfrak{F}_G$ for some finite reflection group G.

There are simple examples of non-centered foliations which satisfy SP.

Terng's Convexity Theorem

Here are some results of Chuu-Lian Terng:

- for any point p of a full isoparametric submanifold $M \subseteq \mathcal{V}$, there is a Coxeter group W which acts in the affine normal space $N_p = N_pM$ (1985);
- the orthogonal projection π_p onto N_p maps the parallel to M manifold M_q through $q \in N_p$ onto the convex hull of Wq (Terng's Convexity Theorem; 1986).

A partial converse to Terng's Convexity Theorem

We say that a submanifold M satisfies the Convexity Theorem (CT for short) if for all $p \in M$

$$\pi_p M = \widehat{M \cap N_p},$$

where the set on the right is a convex polytope with vertices in M which has nonempty interior in N_p.

Theorem

If a centered manifold M satisfy CT, then its normal bundle is globally flat.

CT is true for some non-centered manifolds.

SP for discrete groups of automorphisms of a pointed cone

A convex closed cone C is pointed if $C \cap (-C) = \{0\}$ and generating if $\mathrm{Int}(C) \neq \varnothing$. Let C be such a cone and Γ be a discrete subgroup of $\mathrm{Aut}(C)$ such that

- Γ acts properly on $\mathrm{Int}(C)$,
- it has no finite orbit in the interior of C, and
- $\widehat{O}_x \subset \mathrm{Int}(C)$ for any $x \in \mathrm{Int}(C)$, where $\widehat{O}_x$ is the closed convex hull of O_x.

Theorem

Let C and Γ be as above. Then $\mathfrak{F}_\Gamma$ satisfies SP if and only if Γ is a reflection group.

Isoparametric hypersurfaces
Brief history

- The term “isoparametric hypersurface” is due to Levi-Civita (1937); at that time, $|\nabla F|^2$ and ΔF were called the first and the second differential parameter of F, respectively.
- Segre (1938): an isoparametric hypersurface in $\mathbb{R}^n$ is a plane, or a sphere, or a round cylinder.
- Cartan (1938) classified isoparametric hypersurfaces in hyperbolic spaces, which also turn out to be homogeneous.
- He studied isoparametric hypersurfaces in spheres (1939-1940). In particular, he asked if an isoparametric hypersurfaces in a sphere is homogeneous.
- Ozeki and Takeuchi answered to Cartan’s question in the negative (1975). Their examples were generalized by Ferus, Karcher and Münzner (1981).

Isoparametric submanifolds

There is another equivalent definition of isoparametric submanifolds (Terng, 1985).

Definition

A smooth map $F = (F_1, \dots F_k) : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is called *isoparametric* if F has a regular value and for all i, j

(1) $\langle \nabla F_i, \nabla F_j \rangle$ and ΔF_j are constant on leaves of F for all i, j;

(2) $[\nabla F_i, \nabla F_j]$ is a linear combination of $\nabla F_1, \dots, \nabla F_k$ with coefficients constant on leaves.

The leaves of F define an isoparametric foliation, and the preimage of a regular value is an isoparametric submanifold.

Polar groups

- Definition and classification for compact groups — Dadok, 1985.
- The case of complex reductive groups — Dadok, Kac, 1985.
- A conceptual proof of Dadok's theorem — Eschenburg, Heintze, 1999.
- The restricted holonomy group of a Riemannian manifold is polar according to Berger's classification of holonomy groups.
- The restricted normal holonomy group is polar (Olmos, 1990)

References on SP

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Thank you!