# Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator

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Geometry Days in Novosibirsk, 2013
August 28, 2013

# The setting

- $M$ an oriented compact smooth manifold of dimension $n \geq 2$ (possibly, with boundary).
- A Riemannian metric on $M$: $g = \sum_{j\ell=1}^{n} g_{j\ell}(x)dx^j dx^\ell$.
- A real-valued 1-form on $M$ (magnetic potential)

$$\mathbf{A} = \sum_{j=1}^{n} a_j(x)dx^j.$$

- A real-valued closed 2-form on $M$ (magnetic field):

$$\mathbf{B} = \sum_{j<\ell} b_{j\ell}(x)dx^j \wedge dx^\ell.$$

such that

$$\mathbf{B} = d\mathbf{A} \Longleftrightarrow b_{j\ell} = \frac{\partial a_\ell}{\partial x^j} - \frac{\partial a_j}{\partial x^\ell}.$$

# The operator

The magnetic Schrödinger operator:

$$H^h = (\nabla^h_{\mathbf{A}})^* \nabla^h_{\mathbf{A}} = (ih\, d + \mathbf{A})^* (ih\, d + \mathbf{A}),$$

where $h > 0$ is a semiclassical parameter;

$$\nabla^h_{\mathbf{A}} = ih\, d + \mathbf{A} : C^\infty(M) \to \Omega^1(M);$$

and its adjoint

$$(\nabla^h_{\mathbf{A}})^* = (ih\, d + \mathbf{A})^* : \Omega^1(M) \to C^\infty(M).$$

# Geometrical interpretation

- The magnetic potential $-i\mathbf{A}$ is the connection form for a connection in the trivial complex line bundle

$$\mathcal{L} = M \times \mathbb{C} \to M.$$

- The connection (covariant derivative)

$$d - i\mathbf{A} : C^\infty(M) \to \Omega^1(M).$$

- The magnetic field $-i\mathbf{B}$ is the curvature form of $\nabla_{\mathbf{A}}$:

$$(d - i\mathbf{A})^2 = -i\mathbf{B}.$$

- The operator $H^h$ is the Bochner Laplacian associated with $-\frac{i}{h}\mathbf{A}$.

# Statement of the problem

We consider the Dirichlet realization of $H^h$ in $L^2(M, dx_g)$:

$$D(H^h) = \{u \in W^2(M) : u\,|_{\partial M} = 0\}.$$

This is an unbounded self-adjoint operator in $L^2(M, dx_g)$, which has discrete spectrum: there exists a complete orthogonal system of eigenfunction $u_j \in C^\infty(M), u_j \neq 0$:

$$H^h u_j = \lambda_j(H^h) u_j, \quad u_j\,|_{\partial M} = 0, \quad j = 0, 1, 2, \ldots.$$

## Problem

*To study the asymptotic behavior of the eigenvalues $\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots$ for a fixed $j$ as $h \to 0$ (in the semiclassical limit):*

$$\lambda_j(H^h) \sim ?, \quad h \to 0.$$

# Gauge invariance

The eigenvalues $\lambda_j(H^h)$ depends only on **B**, but not on **A**: if

$$\tilde{\mathbf{A}} = \mathbf{A} + d\phi \Longleftrightarrow \tilde{a}_j = a_j + \frac{\partial \phi}{\partial x^j}, \quad j = 1, 2,$$

then the operators $H^h_{\tilde{\mathbf{A}}}$ and $H^h_{\mathbf{A}}$ are unitarily equivalent

$$H^h_{\tilde{\mathbf{A}}} = e^{-i\phi/h} H^h_{\mathbf{A}} e^{i\phi/h}$$

and therefore

$$\lambda_j(H^h_{\tilde{\mathbf{A}}}) = \lambda_j(H^h_{\mathbf{A}}).$$

# The constant magnetic field on the plane

Consider $\mathbb{R}^2$ and assume

$$g_{jk} = \delta_{jk}, \quad b_{12}\left(= \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right) = b > 0.$$

One can put $a_1(x^1, x^2) = 0, a_2(x^1, x^2) = bx^1$.
The operator $H^h$ takes the form

$$H^h = -h^2 \frac{\partial^2}{\partial (x^1)^2} + (ih\frac{\partial}{\partial x^2} + bx^1)^2.$$

The eigenvalues of infinite multiplicity (Landau levels)

$$\lambda_j(H^h) = (2j+1)hb, \quad j = 0, 1, 2, \ldots.$$

In particular,

$$\lambda_0(H^h) = hb \Longrightarrow (H^h u, u) \geq hb\|u\|^2.$$

# Some history

The starting reference for the spectral analysis of self-adjoint realizations of the magnetic Schrödinger operator is the paper [Avron-Herbst-Simon, 1978]. In particular, this paper shows the role of the module of the magnetic field in dimension 2 and 3.

### Theorem

*Assume* $n = \dim M = 2$*. Write* $\mathbf{B} = b dx_g$ *with* $b \in C^\infty(M)$*. For any* $u \in C_c^\infty(M)$*, we have*

$$(H^h u, u) \geq h \left| \int_M b |u|^2 dx_g \right| .$$

Similar estimate in terms of the module of the magnetic field holds for $n = 3$.
For higher dimensions we have to deal with $\mathrm{Tr}^+(B)$.

# Magnetic wells

Because of the estimate:

$$(H^h u, u) \geq h \left| \int_M b|u|^2 dx_g \right| .$$

it is important for us to consider

$$b_0 = \min_{x \in M} |b(x)|,$$

the set (magnetic wells)

$$U = \{x \in M : |b(x)| = b_0\},$$

and behavior $|b(x)|$ near $U$.

# More history

## R. Montgomery's question (1995)

"Can we hear the locus of the magnetic field" (by analogy with the celebrated question by M. Kac).
By the above estimate, if $n = \dim M = 2$ and $\mathbf{B} = b dx_g$,

$$b_0 = \min_{x \in M} |b(x)| > 0,$$

then

$$\lambda_0(H^h) \geq h b_0.$$

Montgomery observed that if $b$ vanishes non-degenerately along a closed curve $\gamma$ then

$$\lambda_0(H^h) \sim C\, h^{2/3}, \quad h \to 0.$$

# More history

- The case when the magnetic field vanishes at some points for the Dirichlet realization in all dimensions $n \geq 2$ [Montgomery95, Helffer-Mohamed96, Pan-Kwek02, Helffer-Kordyukov07,09, Dombrowski-N.Raymond12].
- The case when the magnetic field never vanishes for the Dirichlet realization in dimension 2 and 3 [Helffer-Morame01, Helffer-Kordyukov11-12-13, N.Raymond-Vu Ngoc13].
- There is a big literature devoted to the spectral analysis of the Neumann realization because of its connection with problems in superconductivity, see the book S. Fournais, B. Helffer. Spectral methods in surface superconductivity. Birkhäuser Boston, Inc., Boston, MA, 2010.

# Assumptions (discrete wells)

Assume $n = \dim M = 2$. Write

$$\mathbf{B} = b dx_g \Longleftrightarrow b = \frac{1}{\sqrt{g}}\left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right), \quad b \in C^\infty(M).$$

Let

$$b_0 = \min_{x\in M} |b(x)|.$$

1. $b_0 > 0$;
2. there exists a unique point $x_0$, which belongs to the interior of $M$, such that $|b(x_0)| = b_0$;
3. for some $k \in \mathbb{N}$ and $C > 0$ we have

$$C^{-1}\, d(x, x_0)^2 \leq |b(x)| - b_0 \leq C\, d(x, x_0)^2$$

for all $x$ in some neighborhood of $x_0$.

# Eigenvalue estimates

Denote

$$a = \mathrm{Tr}\left(\frac{1}{2}\mathrm{Hess}\, b(x_0)\right)^{1/2}, \quad d = \det\left(\frac{1}{2}\mathrm{Hess}\, b(x_0)\right).$$

### Theorem

*Under current assumptions, for any natural $j$,*

$$\lambda_j(H^h) = hb_0 + h^2\left[\frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}\right] + \mathcal{O}(h^{5/2}).$$

# Complete asymptotic expansions

## Theorem

*Under current assumptions, for any natural $j$, there exists a sequence $(\alpha_{j,\ell})_{\ell\in\mathbb{N}}$ such that*

$$\lambda_j(H^h) \sim h\sum_{\ell=0}^{\infty} \alpha_{j,\ell} h^{\frac{\ell}{2}} ,$$

*with*

$$\alpha_{j,0} = b_0, \quad \alpha_{j,1} = 0, \quad \alpha_{j,2} = \frac{2d^{1/2}}{b_0} j + \frac{a^2}{2b_0} .$$

## Remark

Recently, Raymond-Vu Ngoc (2013) proved that in the flat case there is no odd powers in $h$:

$$\lambda_j(H^h) \sim h\sum_{\ell=0}^{\infty} \alpha_{j,2\ell} h^{\ell} .$$

# Approximate eigenfunctions (quasimodes)

The proof of upper estimates is based on a construction of approximate eigenfunctions.

### Theorem

*Under current assumptions for any $j$ and $k$ there exits a sequence $(\mu_{j,k,\ell})_{\ell\in\mathbb{N}}$, and, for any $N$ there exist $\phi^h_{jkN} \in C^\infty_c(U)$, $C_{jk,N} > 0$ and $h_{jk,N} > 0$ such that*

$$(\phi^h_{j_1 k_1 N}, \phi^h_{j_2 k_2 N}) = \delta_{j_1 j_2}\delta_{k_1 k_2} + \mathcal{O}_{j_1,j_2,k_1,k_2}(h)$$

*and, for any $h \in (0, h_{jk,N}]$,*

$$\|H^h \phi^h_{jkN} - \mu^h_{jkN}\phi^h_{jkN}\| \le C_{jk,N} h^{\frac{N+3}{2}} \|\phi^h_{jkN}\|,$$

# Approximate eigenfunctions (quasimodes)

Theorem (continued):

$$\mu^h_{jkN} = h\sum_{\ell=0}^{N} \mu_{j,k,\ell} h^{\frac{\ell}{2}},$$

moreover

$$\mu_{j,k,0} = (2k+1)b_0, \quad \mu_{j,k,1} = 0,$$

and

$$\mu_{j,k,2} = (2j+1)(2k+1)\frac{d^{1/2}}{b_0} + (2k^2+2k+1)\frac{t}{2b_0} + \frac{1}{2}(k^2+k)R(x_0)\,,$$

$$d = \det\left(\frac{1}{2}\mathrm{Hess}_g\, b(x_0)\right), t = \mathrm{Tr}\left(\frac{1}{2}\mathrm{Hess}_g\, b(x_0)\right).$$

$R$ the scalar curvature of the Riemannian metric $g$.

# Geometric interpretations of the coefficients

- The coefficients $\mu_{j,k,\ell}$ in the asymptotic expansions are invariants of the metric $g$ and the connection **A**.
- Under current assumptions, $(M, \mathbf{B})$ is a symplectic manifold.
- The magnetic curvature (Karasev, 2007, deformation quantization of $(M, \mathbf{B})$):

$$\frac{1}{2}\mathrm{Hess}_g\, b(x_0).$$

- The terms

$$(2k+1)hb_0 + \frac{1}{2}h^2\left(k^2+k\right)R$$

have a natural interpretation in terms of eigenvalues of the associated magnetic Schrödinger operator with constant magnetic field on the corresponding Riemann surface of constant curvature $R$ (Landau levels).

# Classical Landau levels: $R = 0$

The Euclidean plane

$$M = \mathbb{R}^2$$

The operator

$$H^h = \left(ih\frac{\partial}{\partial x} - by\right)^2 - h^2 y^2 \frac{\partial^2}{\partial y^2}$$

Eigenvalues

$$\lambda^h(k) = (2k+1)hb, \quad k \in \mathbb{N}.$$

# Hyperbolic Landau levels: $R < 0$

The hyperbolic plane:

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\},$$

endowed with

$$g = \frac{dx^2 + dy^2}{y^2}, \quad (R = -2).$$

The operator (the Maas Laplacian)

$$H^h = y^2 \left( ih\frac{\partial}{\partial x} - by^{-1} \right)^2 - h^2 y^2 \frac{\partial^2}{\partial y^2}.$$

# Hyperbolic Landau levels: $R < 0$

## Descripition of the spectrum (Elstrodt, 1973)

$$\sigma(H^h) = \sigma_{\mathrm{ac}}(H^h) \bigcup \sigma_{\mathrm{disc}}(H^h).$$

Absolutely continuous spectrum:

$$\sigma_{\mathrm{ac}}(H^h) = [h^2 b^2 + \frac{1}{4}, +\infty[\,.$$

Discrete spectrum:

- If $0 \leq hb \leq \frac{1}{2}$, $\sigma_{\mathrm{disc}}(H^h) = \emptyset$.
- If $hb > \frac{1}{2}$, $\sigma_{\mathrm{disc}}(H^h)$ consists of finite number of eigenvalues of infinite multiplicity

$$\lambda^h(k) = (2k+1)hb - h^2\left(k^2 + k\right), \quad k \in \mathbb{N}, k < hb - \frac{1}{2}.$$

# Spherical Landau levels: $R > 0$

- The two-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

- The metric is induced by the standard Euclidean metric in $\mathbb{R}^3$ ($R = 2$).
- The operator is the Bochner-Laplace operator

$$H_n = \nabla_n^* \nabla_n \, .$$

acting on sections of a Hermitian linear bundle $\mathcal{L}_n$ on $S^2$ endowed with a Hermitian connection $\nabla_n$.

# Spherical Landau levels: $R > 0$

- In mathematics, $(\mathcal{L}, \nabla)$ is a prequantization of $S^2$ (Souriau-Kostant geometric quantization). It is a complex linear bundle associated with the Hopf fibration, a $S^1$-principal bundle $S^3 \to S^2$.
- In physics, $(\mathcal{L}_n, \nabla_n)$ is the magnetic Wu-Yang monopole — a natural topological interpretation of the Dirac monopole with magnetic charge $g = nh/2e$.
- Spherical Landau levels:

$$\frac{1}{2}|n|(2k+1) + k^2 + k\,, \quad k \in \mathbb{N}\,,$$

  with multiplicity $|n| + 2k + 1$.
- Take

$$h = \frac{1}{n}, \quad H^h = \frac{1}{n^2}\nabla_n^*\nabla_n\,.$$

# Landau levels and integrability

**Remark**

All three examples of magnetic Schrödinger operators described above are (Darboux) integrable, see:

E. V. Ferapontov, A. P. Veselov. Integrable Schrödinger operators with magnetic fields: factorization method on curved surfaces. J. Math. Phys. **42** (2001), no. 2, p. 590–607.

# The setting

- $\Omega$ is a subset in the flat Euclidean space $\mathbb{R}^3$ with coordinates $X = (X_1, X_2, X_3) = (x, y, z)$.
- $\mathbf{A} = A_1 dX_1 + A_2 dX_2 + A_3 dX_3 \in C^\infty(\bar{\Omega})$ is a magnetic potential.
- $\mathbf{B} = d\mathbf{A} = B_1 dX_2 \wedge dX_3 + B_2 dX_3 \wedge dX_1 + B_3 dX_1 \wedge dX_2$ the corresponding magnetic field:

$$B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \quad B_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \quad B_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$

- $\vec{B} = (B_1, B_2, B_3)$ the vector magnetic field.
- The operator is the Dirichlet realization of the magnetic Schrödinger operator in $\Omega$:

$$H^h = \left(hD_{X_1} - A_1(X)\right)^2 + \left(hD_{X_2} - A_2(X)\right)^2 + \left(hD_{X_3} - A_3(X)\right)^2.$$

# Self-adjointness and discrete spectrum

$$b_0 = \min\{|\vec{B}(X)| \,:\, X \in \Omega\}.$$

Assume:

- There exists a constant $C > 0$ such that for $j = 1, 2, 3$
$$|(\nabla B_j)(X)| \leq C(|\vec{B}(X)| + 1), \quad X \in \Omega.$$
- There exist a (connected) bounded domain $\Omega_1 \subset\subset \Omega$ and a constant $\epsilon_0 > 0$ such that
$$|\vec{B}(X)| \geq b_0 + \epsilon_0, \quad X \notin \Omega_1 \,.$$

## Fact

*The operator $H^h$ is self-adjoint and, for any $\epsilon_1$ with $0 < \epsilon_1 < \epsilon_0$, there exists $h_1 > 0$ such that, for $h \in (0, h_1]$*

$$\sigma(H^h) \cap [0, h(b_0 + \epsilon_1)) \subset \sigma_d(H^h).$$

# Discrete wells

Notation: The eigenvalues of the operator $H^h$ contained in $[0, h(b_0 + \epsilon_0))$ $(= \sigma(H^h) \cap [0, h(b_0 + \epsilon_1)))$:

$$\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots .$$

Assume:

$$b_0 > 0,$$

and there exists a unique minimum $X_0 \in \Omega$ such that $|\vec{B}(X_0)| = b_0$, which is non-degenerate: in some neighborhood of $X_0$

$$C^{-1}|X - X_0|^2 \leq |\vec{B}(X)| - b_0 \leq C|X - X_0|^2 .$$

# Main result

Notation: $d = \det \mathrm{Hess}\, |\vec{B}|(X_0)\,,\, a = \frac{1}{2b_0^2}(\mathrm{Hess}|\vec{B}|\, \vec{B} \cdot \vec{B})(X_0)$.

## Theorem

*Under current assumptions, for any natural m, there exist $C_m > 0$ and $h_m > 0$ such that, for any $h \in (0, h_m]$,*

$$\lambda_m(H^h) \leq hb_0 + h^{3/2} a^{1/2} + h^2 \left[ \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m+1) + \nu \right] + C_m h^{9/4}\,,$$

*where $\nu$ is some explicit constant.*

## Conjecture

$$\lambda_m(H^h) \geq hb_0 + h^{3/2} a^{1/2} + h^2 \left[ \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m+1) + \nu \right] - C_m h^{9/4}\,.$$

# Geometric comments

## Remarks:

- Under current assumptions, $(M, \mathbf{B})$ is a presymplectic manifold, since $\mathbf{B}$ has constant rank: $\operatorname{rank} \mathbf{B} = 2$.
- For $\dim M = 3$, Riemannian geometry is much more complicated — eight geometric models (Thurston).

## Question

To construct the magnetic Schrödinger operators with constant magnetic field on three-dimensional geometric models and compute their spectra.

## Question

Integrable examples of magnetic Schrödinger operators for $n = 3$.

# References


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