THE MATHEMATICS OF PEBBLES: EQUILIBRIA AND MORSE FUNCTIONS

Zsolt Lángi

Department of Geometry, Budapest University of Technology, Hungary

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The presented results are achieved jointly with

Gábor Domokos
Dept. of Mechanics, Materials and Structures, Budapest University of Technology

Gábor Etesi
Dept. of Geometry, Budapest University of Technology

Tímea Szabó
Dept. of Mechanics, Materials and Structures, Budapest University of Technology

The study of the static equilibria of convex solids started with the work of Archimedes. His results were used even in the 18th century naval design (Nowacki, 2002). This topic appears throughout the history of dynamics.

Mathematical aspects of these problems in recent times:

1. existence of a homogeneous, monostatic polyhedron with a small number of facets (Conway, Guy, 1969)
2. no monostatic tetrahedron exists in 3-space (Heppes, 1967)
3. there are monostatic simplices in dimensions $d > 10$ (Dawson, 1985)
4. there are inhomogeneous monostatic tetrahedra in 3-space (Dawson, Finbow, 1999)
5. estimates about the average number of equilibria of convex bodies (Chakerian 1984, Hann, 1993, Hug 1995)
6. a typical convex body has infinitely many equilibria (Zamfirescu, 1995)

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1. Classification of convex solids in terms of equilibria: properties of their Euclidean distance functions as Morse functions
2. The geometric structure of the equilibrium points on pebbles: 'flocks'
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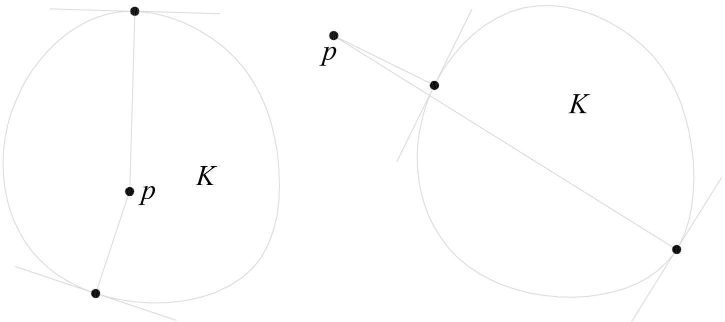
DEFINITION

Let K be a convex body in the Euclidean m-space $\mathbb{R}^m$, and $p \in \mathbb{R}^m$. We say that $q \in \operatorname{bd} K$ is an equilibrium point of K with respect to p, if K has a supporting hyperplane H at q which is perpendicular to the segment $[p, q]$.



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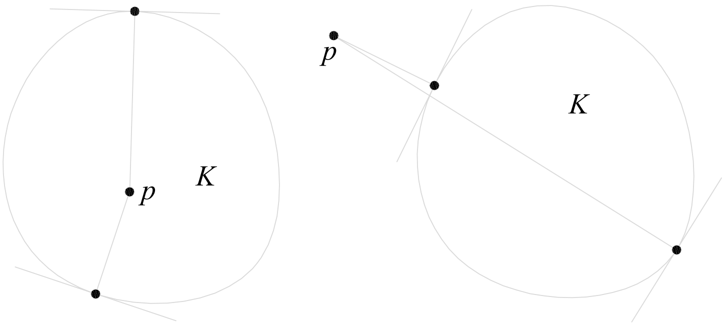

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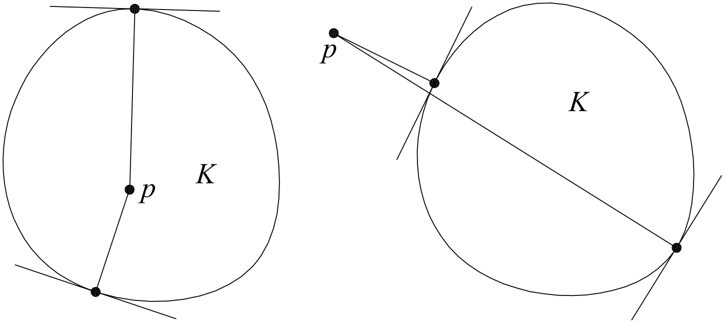

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If $K \subset \mathbb{R}^m$ is a convex body with a C^∞-differentiable boundary, then the equilibria of K with respect to a point $p \in \mathbb{R}^m$ correspond to the critical points of the Euclidean distance function $q \mapsto |p - q|$, $q \in \text{bd}\, K$. If q is an equilibrium point of K with respect to p, and the Hessian of this function is not zero, then the equilibrium is called nondegenerate.

REMARK

If $m = 2$ (or $m = 3$), a nondegenerate equilibrium can be stable or unstable (in $m = 3$ also saddle) points, depending the number of the negative eigenvalues of the Hessian.

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In most parts, we deal only with convex bodies in $\mathbb{R}^2$ or $\mathbb{R}^3$, with C^∞-differentiable boundaries. We always assume that K has finitely many equilibrium ponts, all of which are nondegenerate.

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Theorem (special case of Poincaré-Hopf Theorem)

Let $K \subset \mathbb{R}^m$ be a convex body with $m = 2$ or 3, and let $p \in \operatorname{int} K$. Let S, U and H denote the number of stable, unstable and saddle points of K with respect to p, respectively. Then,

$$\begin{aligned} S - U &= 0 \quad \text{if } m = 2; \\ S - H + U &= 2 \quad \text{if } m = 3. \end{aligned}$$

Definition

The centre of gravity of a convex body $K \subset \mathbb{R}^m$ is defined as

$$\frac{1}{\operatorname{vol}(K)} \left(\int_{x \in K} x \, \mathrm{d}V \right)$$

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The family of 3-dimensional convex bodies with S stable and U unstable points with respect to their centres of gravity is denoted by $\{S, U\}$. In the plane, the class $\{S\}$ is defined similarly.(Primary classification)

Question

Which of the classes $\{S\}$ and $\{S, U\}$ are empty, if $S, U \geq 1$?

Theorem (Domokos, Ruina, Papadopoulos, 1994)

Every plane convex body has at least two stable and two unstable points with respect to its centre of gravity; i.e. the class $\{1\}$ is empty.

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No class $\{S\}$, where $S \geq 2$ is empty. Examples: ellipses, smoothened regular polygons.

CONJECTURE (ARNOLD, 1995)

There exist a mono-monostatic solid; that is, the class $\{1, 1\}$ is not empty.

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Arnold's conjecture is true.

The body they constructed is called 'Gömböc'.
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1. Classification of convex solids

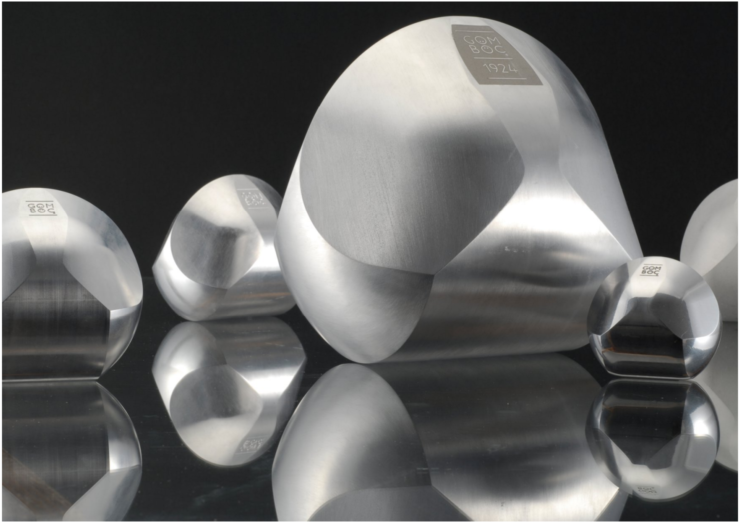

FIGURE: Gömböc #1924 in the National Bank of Hungary, 2008

FIGURE: Arnold receives Gömböc #1, 2007

FIGURE: Snorre H. Christiansen receives a Gömböc as part of the Stephen Smale Prize, 2011

Figure: Gömböc in the centre of the Hungarian Pavilion on the World Expo in Shanghai, 2010

FIGURE: A turtle shell resembling a Gömböc

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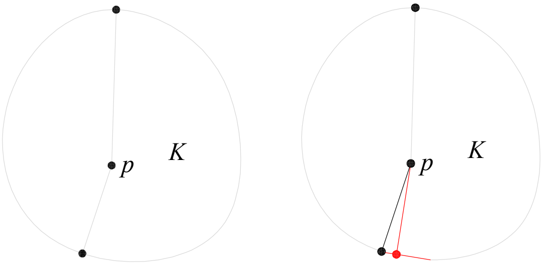

FIGURE: Creating a new stable point near an existing stable point

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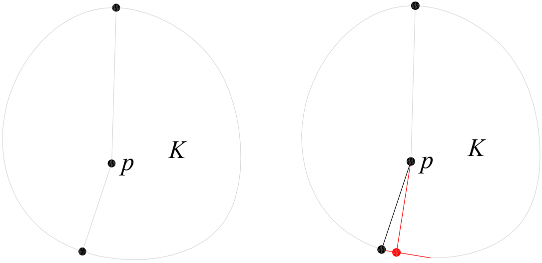

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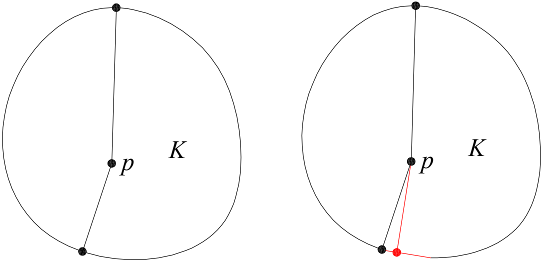

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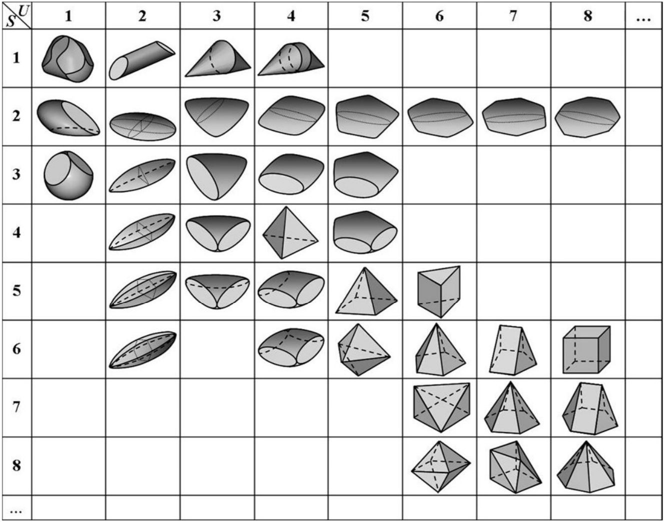


FIGURE: Primary classification scheme

Secondary classification scheme

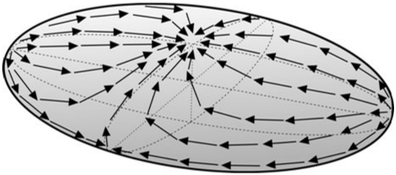


FIGURE: Gradient vector field

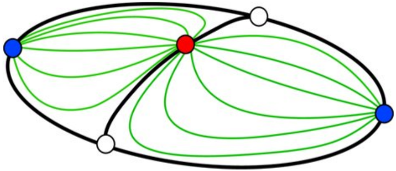


FIGURE: Heteroclinic orbits

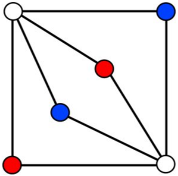


FIGURE: Morse-Smale graph in the plane

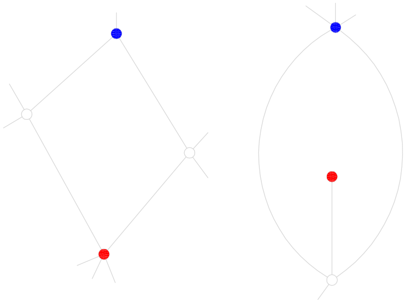
RED: stable point

BLUE: unstable point

WHITE: saddle point

DEFINITION

A 3-colored quadrangulation of $\mathbb{S}^2$ is a drawing of a graph on $\mathbb{S}^2$, possibly with multiple edges but without loops, such that each vertex is labelled with one of the symbols, say, *S*, *U* and *H*, and each face is bounded by a closed walk consisting of vertices labelled with *U*, *H*, *S* and *H* in this order.



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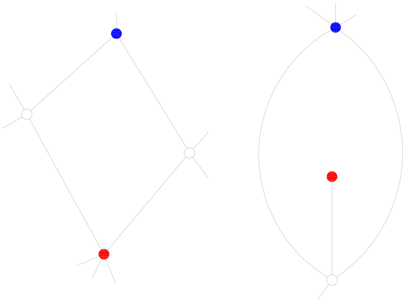

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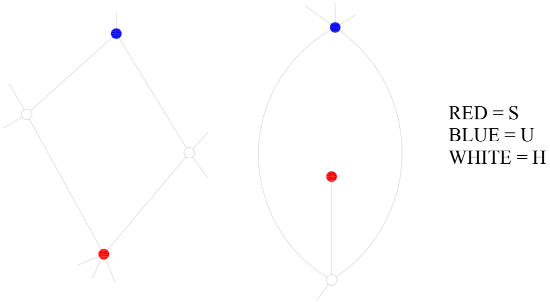

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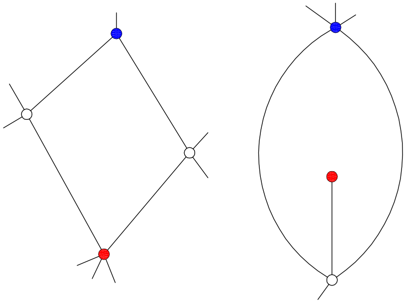

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The combinatorial properties of Morse-Smale graphs (Edelsbrunner, Harer, Zomorodian, 2003):

- it is a 3-colored quadrangulation of $\mathbb{S}^2$ such that the colors correspond to the minima, maxima and saddle points of the Morse function;
- the degree of every saddle point is 4;
- $\#S - \#H + \#U = 2$.

In the literature, the graph G_0, with two vertices (one S, one U) and no edge, is meant to belong to this class by agreement.

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In the literature, the graph G_0, with two vertices (one S, one U) and no edge, is meant to belong to this class by agreement.

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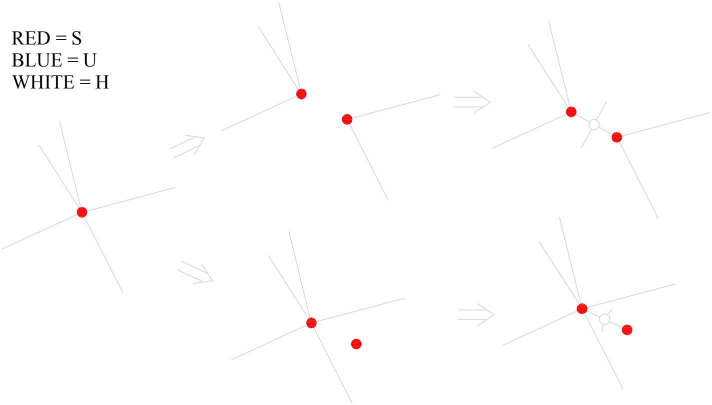

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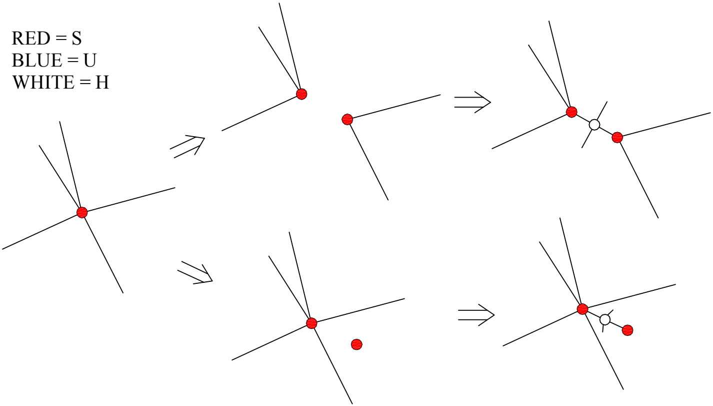

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The structure of the proof:

1) Given a graph G satisfying the conditions, we reduce it to G_0 with a sequence of face contractions. Then G can be generated from G_0 by the sequence of the corresponding vertex splittings.

2) We realize these vertex splittings geometrically, starting with a mono-monostatic body (Gömböc).

Geometric part: given a convex body K and any vertex splitting of the Morse-Smale graph of K, there is a convex body K' such that its Morse-Smale graph is the split one.

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Other classifications systems:

DEFINITION (NICOLAESCU, 2008)

Let $f : \mathbb{S}^2 \to \mathbb{R}$ be a Morse function, with n, nondegenerate critical points, no two of which lie on the same level. A slicing of f is an increasing sequence of numbers $a_0 < a_1 < \ldots < a_n$ such that for any i, (a_{i-1}, a_i) contains exactly one critical value of f.

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Two Morse functions f and g are called geometrically equivalent, if there is an orientation-preserving diffeomorphism $R : \mathbb{S}^2 \to \mathbb{S}^2$ and an orientation-preserving diffeomorphism $L : \mathbb{R} \to \mathbb{R}$ such that

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QUESTION (DOMOKOS, ETESI, LÁNGI, 2013)

Is it true that every equivalence class of $\sim_g$, $\sim_t$ or $\sim_h$ can be represented by the Euclidean distance function of a convex body from its centre of gravity?

A natural choice of studying convex bodies is to study pebbles.

Observation (Domokos, Sipos, Szabó, Várkonyi, 2010): equilibrium points always appear in groups (called flocks).

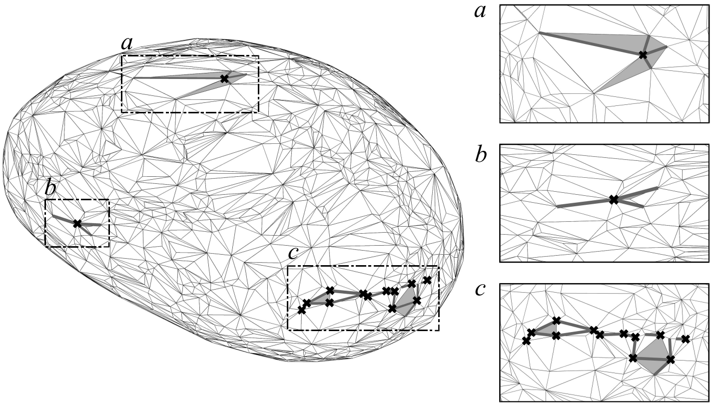

FIGURE: Flocks on a pebble

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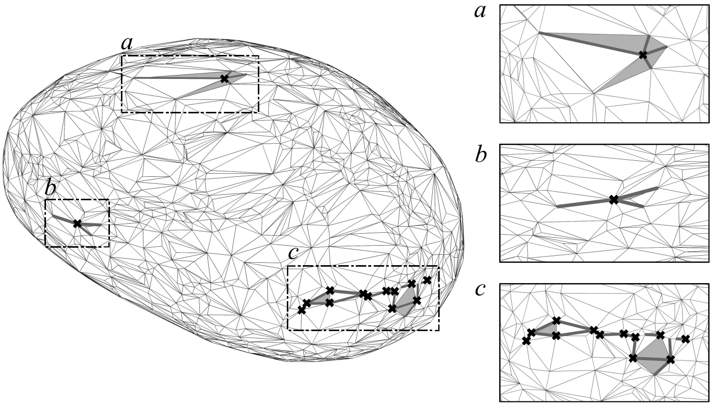

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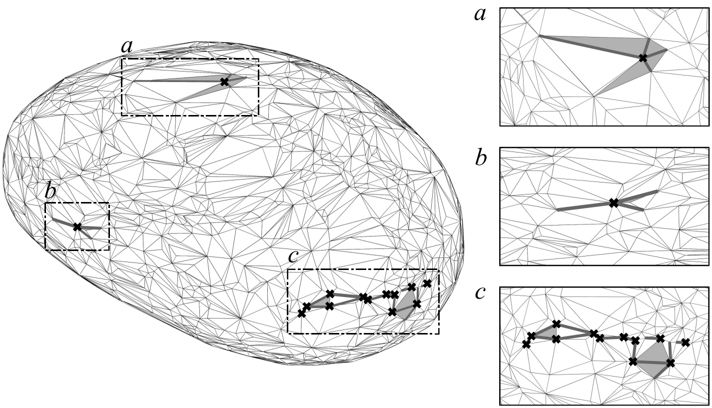

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Observation (Domokos, Sipos, Szabó, Várkonyi, 2010)

The boundary of the convex hull of a pebble is a many-faceted polyhedron, and not smooth.

Remark (Domokos, Sipos, Szabó, Várkonyi, 2010)

Explanation for flocks: there are two levels of equilibrium points.

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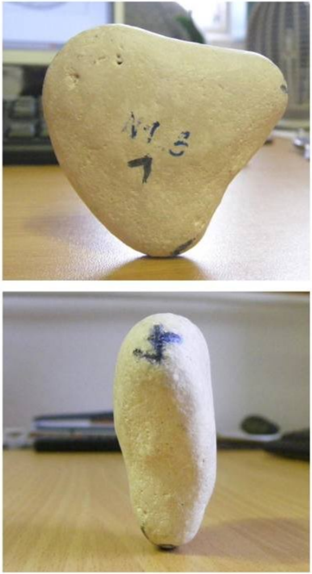
This explains the existence of rocking stones (Domokos, Sipos, Szabó, 2011).



Plan: Let us consider a smooth convex surface with only one equilibrium point, and approximate it with a many-faceted polyhedral surface. Examine what happens to the number of equilibria.

Let $r : [0,1] \times [0,1] \to \mathbb{R}^3$ be a C^3-differentiable convex surface with only one, nondegenerate equilibrium point $p = r(u_0, v_0)$ with respect to the origin o. Let $|p| = \rho$, and κ_1, κ_2 be the two principal curvatures of r at p.

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The point p is nondegenerate if, and only if $\kappa_1, \kappa_2 \neq \frac{1}{\rho}$.

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For any positive integer n, let $q_{i,j} = r\left(\frac{i}{n}, \frac{j}{n}\right)$.

Let P_n be the polyhedral surface defined in the following way:
1) The vertices of P_n are the points $q_{i,j}$.
2) The faces of P_n are the faces of $\text{conv}\{o, q_{i,j}, q_{i+1,j}, q_{i,j+1}, q_{i+1,j+1}\}$ that does not contain o.

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For any $K > 0$, let $S^n(K)$, $U^n(K)$ and $H^n(K)$ denote the number of stable, unstable and saddle points of P_n, with respect to the origin, that can be connected to the face of P_n, intersecting the segment $[o, p]$, with a curve in P^n passing through at most K faces.

THEOREM (DOMOKOS, LÁNGI, SZABÓ, 2012)

Let $K = K(\rho, \kappa_1, \kappa_2)$ be sufficiently large. Then there is a value n_0 such that for every $n > n_0$, we have

$$\left| U^n(K) - \frac{\rho^2 \kappa_1 \kappa_2}{|(1-\kappa_1\rho)(1-\kappa_2\rho)|} \right| \leq \mathrm{Err}_U,$$

$$\left| S^n(K) - \frac{1}{|(1-\kappa_1\rho)(1-\kappa_2\rho)|} \right| \leq \mathrm{Err}_S,$$

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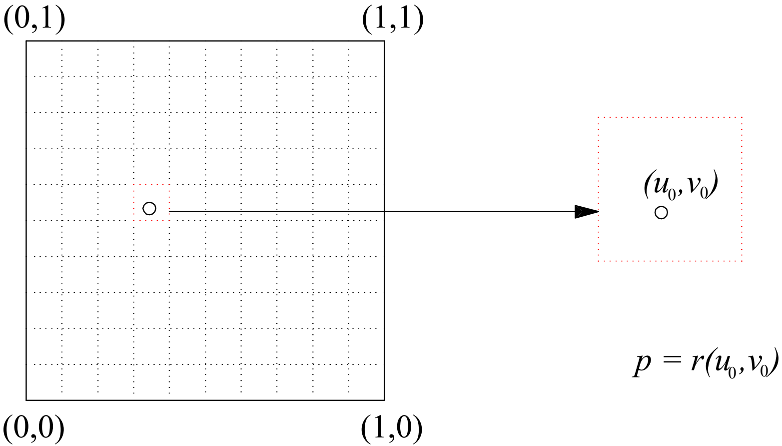

FIGURE: The relative position of (u_0, v_o) within its grid cell

THEOREM (DOMOKOS, LÁNGI, SZABÓ, 2012)

Assume that the parameters u_0, v_0 of $p = r(u_0, v_0)$ are irrational and linearly independent over $\mathbb{Q}$. If $K = K(\rho, \kappa_1, \kappa_2)$ is sufficiently large, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} U^k(K)}{n} = \frac{\kappa_1 \kappa_2 \rho^2}{|(1-\kappa_1\rho)(1-\kappa_2\rho)|},$$

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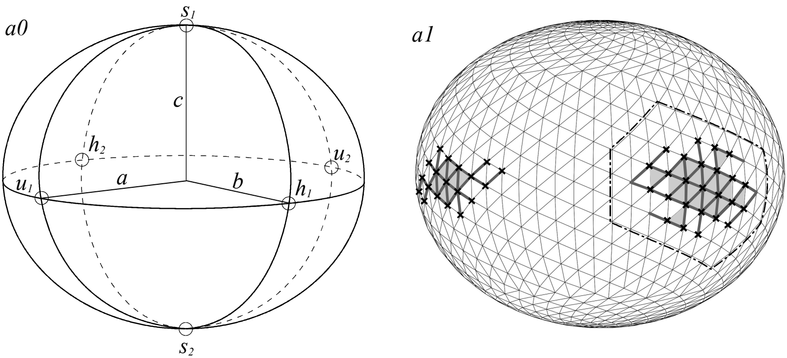

FIGURE: Flocks on an ellipsoid using uniform discretization

REMARK

Instead of uniform discretization, perhaps a stochastic approach describes the phenomenon better.

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Let us choose n points on the surface independently, using uniform distribution. What is the expected value of the number of equilibrium points on their convex hull?

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Given a convex solid K, what is the minimal volume of a truncation, relative to the volume of K, that changes the equilibrium class of the solid?

We call this quantity the robustness of K.

Motivation: in physical abrasion processes small materials are chipped off from the solid, from time to time.

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DEFINITION

Let $K \subset \mathbb{R}^2$ be a plane convex body with piecewise smooth (C^∞-class) boundary, and let $p \in \operatorname{int} K$. Let q be an equilibrium point of K with respect to p. We say that q is nondegenerate, provided that

- if q is a smooth point of $\operatorname{bd} K$, as in the previous part;
- if q is not a smooth point, then if the angles of both one-sided tangent half lines of K at q, with the segment $[p, q]$, are acute.

A nonsmooth, nondegenerate equilibrium point is called unstable.

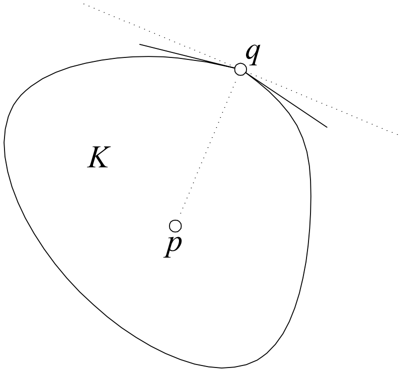

FIGURE: Nondegenerate, nonsmooth equilibrium point

We extend the definition of $\{S\}$ to plane convex bodies with piecewise smooth boundary, and the definition of $\{S, U\}$ also for convex polyhedra.

DEFINITION

Let $K \in \{S\}$, and let $\mathfrak{F}(K)$ denote the family of convex subsets of K with strictly less than S stable points with respect to their centres of gravity. Then the robustness of K is

$$\rho(K) = \frac{\inf\{\text{area}(K \setminus K') : K' \in \mathfrak{F}(K)\}}{\text{area}(K)}.$$

If the set in the numerator is empty, we let $\rho(K) = 1$.

We can define $\rho(K)$ in $\mathbb{R}^3$ similarly.

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Let $\mathcal{K}_2$ denote the family of plane convex bodies with piecewise smooth boundary.

Definition

Let $K \in \mathcal{K}_2$ have S stable points with respect to some point $p \in \operatorname{int} K$. Let $\mathfrak{F}(K,p)$ denote the family of convex subsets of K that have strictly less than S stable points with respect to p. Then the external robustness of K with respect to p is

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$$\rho_{in}(K,p) = \frac{\inf\{|q-p| : q \notin R(K,p)\}}{\operatorname{perim}(K)}.$$

We can define $\rho_{in}(K,p)$ in $\mathbb{R}^3$ similarly, if we replace perim K by $\sqrt{\operatorname{surf} K}$.

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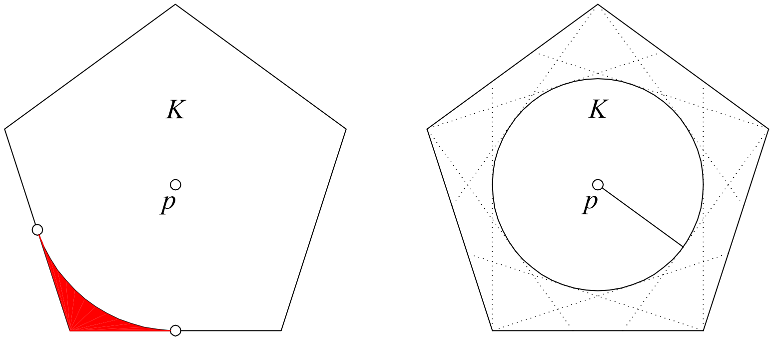

FIGURE: External, internal robustness of a regular pentagon with respect to its centre

Theorem (Domokos, Lángi, 2013)

Let $K \in \mathcal{K}_2$ be a plane convex body with $S \geq 3$ stable points with respect to some point $p \in \operatorname{int} K$. Then

$$\rho_{ex}(K,p) \leq \frac{\tan\frac{\pi}{S} - \frac{\pi}{S}}{S \tan\frac{\pi}{S}},$$

with equality if, and only if K is a regular S-gon and p is its centre.

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Let $K \in \mathcal{K}_2$ be a plane convex body with $S \geq 3$ stable points with respect to some $p \in \operatorname{int} K$. Then

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Theorem (Domokos, Lángi, 2013)

Let P be a regular polyhedron with S faces, U vertices and $H = S + U - 2$ edges, and with p as its centre. Let P' be a convex polyhedron with S faces, U vertices and H edges, each containing an equilibrium point with respect to some $q \in \operatorname{int} P'$. Then

$$\rho_{in}(P', q) \leq \rho_{in}(P, p),$$

with equality if, and only if P' is a similar copy of P, with q as its centre.

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The previous theorem is false with the weaker assumption that P' has S stable, U unstable and H saddle points, but not necessarily S faces, H edges and U vertices.

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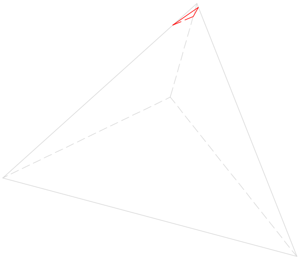
Let P be a regular tetrahedron with centre o. Truncate P near a vertex, with a plane almost parallel to a face of P such that the truncated part does not intersect the incircle of any face of P. Denote the truncated polyhedron by P'. Then, for P and P', the numerators in the definition of $\rho_{in}(P, o)$ are equal, but $\mathrm{surf}(P') < \mathrm{surf}(P)$, which yields $\rho_{in}(P, o) < \rho_{in}(P', o)$.



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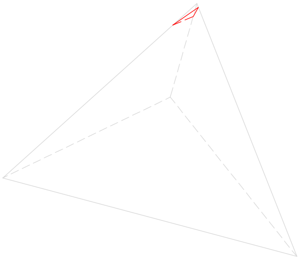

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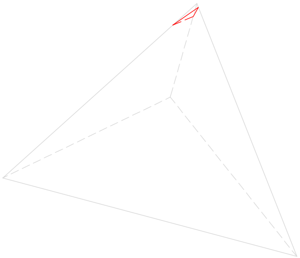

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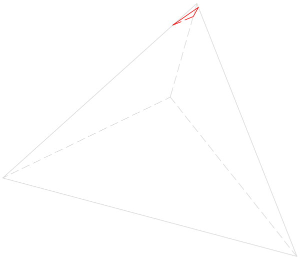

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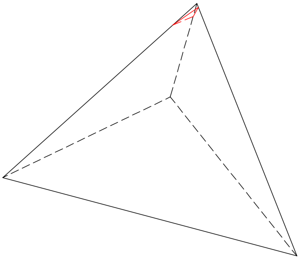

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$$\begin{aligned}\rho_i &= \sup\{\rho(K) : K \in \{i\}\}\\ \rho_{i,j} &= \sup\{\rho(K) : K \in \{i,j\}\}\end{aligned}$$

Remark

Since $\{1\} = \emptyset$, ρ_1 does not exist. By definition, $\rho_{1,1} = 1$.

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In $\mathbb{R}^3$, one may define partial robustness (*S*-robustness and *U*-robustness): the minimal relative volume of the truncation that decreases the number of stable/unstable points. We may define $\rho^s_{i,j}$ and $\rho^u_{i,j}$ accordingly.

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3. Robustness of equilibria

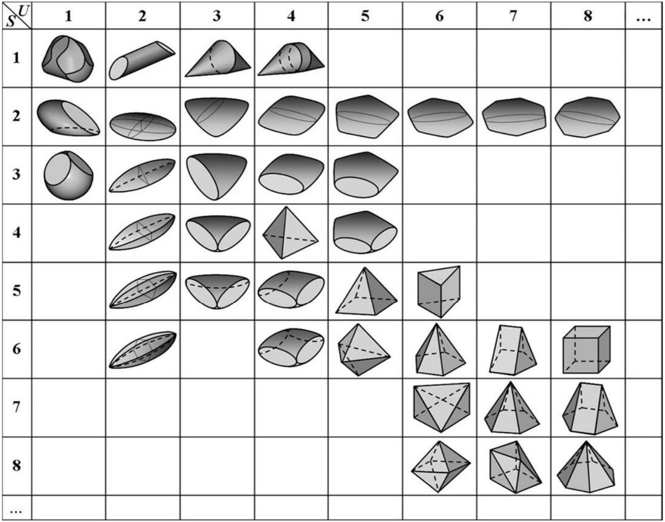

FIGURE: $\{S, U\}$ classes

Remark (Domokos, Sipos, Szabó, Várkonyi, 2006, 2010)

According to field experiments, in coastal regions, the ratio of monostatic pebbles is found to be below 0.1%.

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If $i \geq k$ and $j \geq l$ then $\rho_{i,j} \leq \rho_{k,l}$.

REFERENCES

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Thank You

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