

Recent Progress in Map Enumeration

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new results done in collaboration with A. Mednykh
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Map Enumeration Problem

Maps Map is a 2-cell decomposition of a surface

Category OrMaps Maps on orientable surfaces together with orientation-preserving homomorphisms

Category Maps Maps on surfaces (possibly with non-empty boundary) together with homomorphisms

Enumeration problem: Given property \mathcal{P} determine a function $N_{\mathcal{P}}(e)$ counting the number of maps in \mathcal{P} with e edges (e edges and v vertices).

Problems considered in this talk

\mathcal{P} is the set of all ORMAPS,

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\mathcal{P} is the set of ORMAPS of given genus,

\mathcal{P} is the set of MAPS on a fixed S ,

Why to do Map Enumeration?

- ① **External motivation:** coming from chemistry, **statistical physics**, theory of strings, biology ...

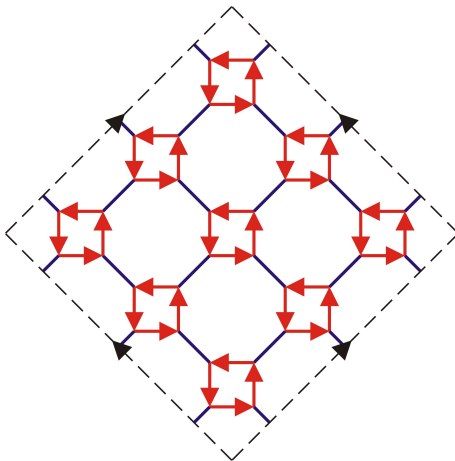
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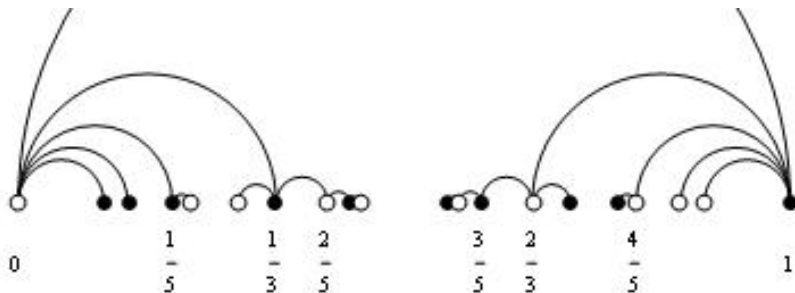
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- ③ **Internal:** Enumeration is a synonymum of classification, map generation, asymptotic behavior of maps, learning more on maps, **Chiral versus Reflexible**,...

Feynmann diagrams



Maps endowed with geometry: Geometry is a well-organized combinatorics



Universal hypermap in the upper-half plane

Theorem of Belyj and Grothendieck programme

A Belyj function is a meromorphic function $X \rightarrow \Sigma$ with no critical values outside $\{0, 1, \infty\}$.

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Grothendieck programme: Dessign d'Enfants

!!! The **absolute Galois group** acts faithfully on maps

It acts faithfully even on tree-like maps.

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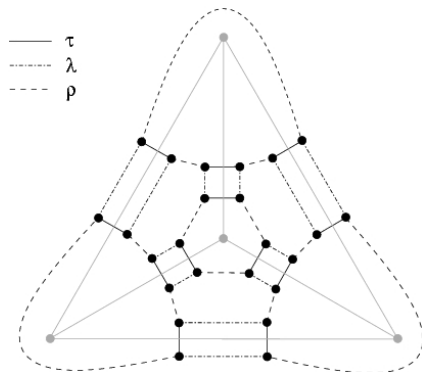
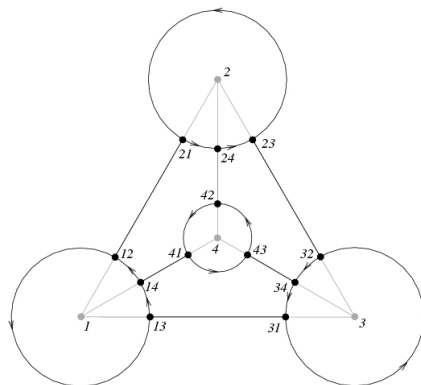
Problem Determine the orbits of the action on trees of order n . **How to be sure that we have all?**

Combinatorial description of maps

- ORMAPS: $(D; R, L)$, $R, L \in \text{Sym}(D)$, $L^2 = 1$, $\langle R, L \rangle$ is transitive on D ,
- MAPS: $(F; r, \ell, t)$, $r, \ell, t \in \text{Sym}(F)$,
 $r^2 = \ell^2 = t^2 = (lt)^2 = 1$, $\langle r, \ell, t \rangle$ is transitive on F ,
- ORDINARY MAPS: L, r, t, ℓ are fixed-point-free, but the category is not closed under taking quotients!
- Combinatorial maps (schemes) describe topological maps up to isotopy.

!!! Map = a particular action diagram of a permutation group = a particular 3-edge coloured cubic graph !!!

Combinatorial maps



Maps and subgroups of $\mathbb{Z} * \mathbb{Z}_2$, $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$

Orientable case (sensed maps)

$$G = \mathbb{Z} * \mathbb{Z}_2 = \langle r, \ell \mid \ell^2 = 1 \rangle$$

given map $(D; R, L)$ the mapping $r \mapsto R, \ell \mapsto L$
extends to a *transitive homomorphism* $G \rightarrow \text{Sym}(D)$.

G acts on D , a stabilizer H of this action is a subgroup of G of index $|D|$.

Vice-versa, given $H \leq G$ of finite index, we can set
 $D = \{xH \mid x \in G\}$, and define $R(xH) = rxH, L(xH) = \ell xH$.

Maps and subgroups of $\mathbb{Z} * \mathbb{Z}_2$, $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$

OBSERVATIONS:

- the underlying surface is **compact** = H is of **finite index**,

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- map is **connected** = the respective homomorphism $G \rightarrow \text{Sym}(D)$ is **transitive**,
- the subgroups that correspond to ORMAPS are free.

Labelled and Rooted Maps

Rooted ormap An ormap with one dart distinguished as a root,

Rooted map A map with one flag distinguished as a root, on an orientable closed surface it is the same!!!

Labelled ormap An ormap with all darts distinguished,

Labelled map A map with all darts distinguished,

Dictionary: equivalences on maps and subgroups

Rooted ormap = a torsion-free subgroup of $\mathbb{Z} * \mathbb{Z}_2$ of finite index,

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Isoclass of an ormap = conjugacy class of a torsion-free subgroup of $\mathbb{Z} * \mathbb{Z}_2$,

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Decomposition of the enumeration problem

WHY ROOTED MAPS? WHY LABELLED MAPS?

Given class of maps the enumeration problem decomposes into three subproblems:

- 1 enumeration of rooted maps,
- 2 enumeration of unrooted maps,
- 3 asymptotic analysis (usually a formula is not closed).

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!!! **Action of $\text{Aut}(M)$ is semiregular** !!! Rooted ormaps and maps have trivial automorphism group!

Example: Counting Group Homomorphisms

GENERAL SCHEME

Theorem

Let $G = \langle x_1, x_2, \dots, x_r \rangle$ be a group. The number of all homomorphisms B_n $G \rightarrow S_n$ and the number T_n of such transitive homomorphisms are related by

$$B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} T_{n-i} B_i,$$

$$\sum_{k=1}^{\infty} \frac{T_k}{k!} z^k = \log \sum_{k=1}^{\infty} \frac{B_k}{k!} z^k.$$

Stanley Vol. II, 1999, **Hurwitz** paper 1891, **M. Hall** in 1949.

Example: Labelled Sensed Maps Regardless of Genus

!!! **Labelled sensed map** (D, R, L) based on n darts $1, 2, \dots, n$ is a **transitive homomorphism** $\mathbb{Z} * \mathbb{Z}_2 \rightarrow S_n$

Darts $D = \{1, 2, \dots, n\}$

rotation R any permutation - $n! = (2e)!$ choices,

dart-reversing involution number of f. p. free involutions is $\frac{(2e)!}{2^e e!}$

Hence we have a direct formula for $B_{2e} = (2e)! \frac{(2e)!}{2^e e!}$ and we can apply the above recursive formula to derive T_{2e} .

!!! All actions is much easier to enumerate than transitive ones!!!

!!! Number of labelled maps = $(n - 1)!$ number of rooted ones.

Example: Rooted Sensed Maps Regardless of Genus

Theorem

The number of rooted orientable maps $R^+(e)$ with e edges is given by the following equation

$$\sum_{e \geq 1} \frac{R^+(e)}{e} 2^{e-1} u^e = \log\left(\sum_{e \geq 0} \frac{(2e)!}{e!} u^e\right).$$

Proof. $T_{2e} = R^+(e)(2e - 1)!$ and formula for T_{2e} .

- (a) non-elementary proof by **D.M. Jackson and T.J. Visentin** in TAMS 332 (1990)
- (b) another different proof by **D. Arques and J.S. Bacuteraud** DM 215 (2000)

Example: Rooted Maps Regardless of Genus

Labelled map on n flags is a transitive homomorphism $G \rightarrow S_n$, where

$$G = \langle x, y, z \mid x^2 = y^2 = z^2 = (yz)^2 = 1 \rangle.$$

There are no essentially new ideas but technically it is harder.

Theorem

The number of rooted maps (orientable or not) $R(e)$ with e edges is given by the following equation

$$\sum_{e \geq 1} \frac{R(e)}{e} 4^{2e-1} u^e = \log \left(\sum_{e \geq 0} \frac{(4e)!}{(2e)! e!} u^e \right).$$

Remark. The number $R^-(e)$ of rooted non-orientable maps with e edges is given by the formula $R^-(e) = R(e) - R^+(e)$.

Rooted maps of given genus, $g = 0$ and $g = 1$

Tutte 1963: the number of rooted planar maps with e edges is

$$\mathcal{N}_0(e) = \frac{2(2e)!3^e}{e!(e+2)!}$$

D. Arques 1987, Bender and Canfield 1988: Rooted toroidal maps

$$\mathcal{N}_1(e) = \sum_{k=0}^{e-2} 2^{e-3-k} (3^{e-1} - 3^k) \binom{e+k}{k}.$$

Rooted maps of genus $g > 1$

D.Arqués and A. Giorgetti, JCT B, 1999

Theorem

For any g the ordinary generating function $Q_g(z) = \sum_{n \geq 0} N_g(n)z^n$ counting rooted maps on S_g by number of edges can be written as

$$Q_g(z) = z^{2g}(1 - 3m)^{-2}(1 - 2m)^{4-5g}(1 - 6m)^{3-5g}P_g(m),$$

where $m = \frac{1 - \sqrt{1 - 12z}}{6}$ and $P_g(m)$ is an integer polynomial of degree $\leq 6g - 6$.

It says: **For a fixed $g > 1$ the generating function is known up to finitely many coefficients of a polynomial.**

Fast recursive algorithm for computing first numbers?

To get a formula one needs to determine the coefficients of $P_g(m)$, this can be done,

- 1 first by generating the first $f(g)$ coefficients of $Q_g(z)$ independently.
- 2 secondly computing the polynomial $P_g(z)$ employing the determined numbers and solving a system of recursions.

Giorgetti, Walsh and **Mednykh**: A recursive algorithm computing first numbers in the sequence, they determined the formulae up to genus 11

Problem: How to enumerate ISOCLASSES of Maps?

Long time only the result for spherical ORMAPS **Liskovets, Wormald (1981)**:

$$\Theta_0(e) = \frac{1}{2e} \left(\alpha(e) + \sum_{\substack{d|e \\ d < e}} \phi(e/d) \binom{d+2}{2} \mathcal{N}_0(d) \right)$$

$\alpha(e) = \mathcal{N}_0(e) + \binom{e}{2} \mathcal{N}_0(e/2-1)$ for e even,

$\alpha(e) = \mathcal{N}_0(e) + e \left(\frac{e-1}{2} + 2 \right) \mathcal{N}_0((e-1)/2)$ for e odd.

$\mathcal{N}_g(e)$ denotes the number of **rooted** maps of genus $g \geq 0$ with e edges

Mednykh's lemma: From subgroups to conjugacy classes

RECALL the dictionary: ISOCLASS of (OR)MAPS = **conjugacy class of subgroups** in the universal group

Theorem (Mednykh)

Let Γ be a finitely generated group. Let \mathcal{P} be a set of subgroups of Γ closed under conjugation. Then the number of conjugacy classes of subgroups of index n in \mathcal{P} is given by the formula

$$N_{\Gamma}^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} \sum_{\substack{K < \Gamma \\ [\Gamma:K]=m}} \text{Epi}_{\mathcal{P}}(K, Z_{\ell}).$$

$\text{Epi}_{\mathcal{P}}(K, Z_{\ell})$ - number of order preserving epimorphisms $K \rightarrow Z_{\ell}$ s.t. the kernel $\in \mathcal{P}$.

Unrooted ormaps on S_g with $n = 2e$ darts

The universal group is $\Gamma = \Delta(\infty, \infty, 2) = \langle x, y \mid y^2 = 1 \rangle$ and \mathcal{P} is a set of subgroups of genus g and index n then

$$\sum_{\substack{K < \Gamma \\ [\Gamma:K]=m}} \text{Epi}_{\mathcal{P}}(K, Z_\ell) = \sum_{O \in \text{Orb}(S/Z_\ell)} h_O(m) \text{Epi}_0(\pi_1(O), Z_\ell),$$

where the second sum runs through all admissible cyclic orbifolds S_g/Z_ℓ . Hence the number of unrooted maps of genus g with $e = \frac{n}{2}$ edges is

$$N_{\Gamma}^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} \sum_{O \in \text{Orb}(S/Z_\ell)} h_O(m) \text{Epi}_0(\pi_1(O), Z_\ell).$$

Problem A: $h_O(m) = ?$, **Problem B:** $\text{Epi}_0(\pi_1(O), Z_\ell) = ?$.

Axioms for ormaps orbifolds

$h_O(m)$ = the number of **rooted maps on a cyclic orbifold**
 $O(g; m_1, m_2, \dots, m_r)$

- (P1) if $x \in B$ then x is either an internal point of a face, or a vertex, or an end-point of a semiedge which is not a vertex,
- (P2) each face contains at most one branch point,
- (P3) the branch index of x lying at the free end of a semiedge is two.

Counting Rooted Ormaps on closed orientable orbifolds

Theorem (N+Mednykh)

Let $O = O[g; 2^{q_2}, \dots, \ell^{q_\ell}]$ be an orbifold, $q_i \geq 0$ for $i = 2, \dots, \ell$. Then the number of rooted maps $\nu_O(m)$ with m darts on the orbifold O is

$$\nu_O(m) = \sum_{s=0}^{q_2} \binom{m}{s} \binom{\frac{m-s}{2} + 2 - 2g}{q_2 - s, q_3, \dots, q_\ell} \mathcal{N}_g((m-s)/2),$$

with a convention that $\mathcal{N}_g(n) = 0$ if n is not an integer, $\mathcal{N}_g(n)$ is the number of rooted ordinary maps of genus g with n edges.

Note: The multinomial counts the number of distributions of branch points between the vertices, faces and free-ends of semiedges!

Fundamental group of cyclic 2-dim. orbifolds

To determine $\text{Epi}_0(K, \mathbb{Z}_\ell)$, we need to know the structure of K .
Answer: $K = \pi_1(O)$ is an F-group

$$\begin{aligned} \pi_1(O) = F[\gamma; m_1, m_2, \dots, m_r] = \\ \langle a_1, b_1, a_2, b_2, \dots, a_\gamma, b_\gamma, x_1, \dots, x_r \mid \\ \prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^r x_j = 1, x_1^{m_1} = \dots x_r^{m_r} = 1 \rangle. \end{aligned}$$

Number of epimorphisms $\pi_1(O) \rightarrow Z_\ell$

Theorem (N.+Mednykh)

Let $\Gamma = F[g; m_1, \dots, m_r]$ be an F -group of signature $(g; m_1, \dots, m_r)$ and $m = \text{lcm}(m_1, \dots, m_r)$, $m|\ell$. Then the number of order-preserving epimorphisms of the group Γ onto a cyclic group Z_ℓ is given by the formula

$$\text{Epi}_0(\Gamma, Z_\ell) = m^{2g} \phi_{2g}(\ell/m) E(m_1, m_2, \dots, m_r),$$

where

$$E(m_1, m_2, \dots, m_r) = \frac{1}{m} \sum_{k=1}^m \Phi(k, m_1) \cdot \Phi(k, m_2) \dots \Phi(k, m_r).$$

In particular, if $\Gamma = F[g; \emptyset] = F[g; 1]$ is a surface group of genus g we have

$$\text{Epi}_0(\Gamma, Z_\ell) = \phi_{2g}(\ell).$$

Epi enumeration - main idea

$$|\text{Hom}(G, \mathbb{Z}_\ell)| = \sum_{d|\ell} |\text{Epi}(G, \mathbb{Z}_d)|,$$

Jordan function: $\varphi_p(\ell) = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^p$

VonSterneck function:

$$\Phi(x, n) = \frac{\phi(n)}{\phi\left(\frac{n}{(x, n)}\right)} \mu\left(\frac{n}{(x, n)}\right) = \sum_{\substack{1 \leq k \leq n \\ (k, n)=1}} \exp\left(\frac{2ikx}{n}\right).$$

Reduction to homology group

For enumeration of sensed maps of given genus $\pi_1(O)$ is an F-group with relations:

$$\prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^r e_j = 1, e_1^{m_1} = \dots e_r^{m_r} = 1.$$

Reduction to epimorphisms from the homology group

$$H_1(O) \rightarrow \mathbb{Z}_d$$

$$\{(a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_r) \in \mathbb{Z}_d^{2g+r} :$$

$$x_1 + \dots + x_r = 0 \pmod{d}, (x_1, d) = d_1, \dots, (x_r, d) = d_r\},$$

Orbicyclic function

The function

$$E(m_1, m_2, \dots, m_r) = \frac{1}{m} \sum_{k=1}^m \Phi(k, m_1) \cdot \Phi(k, m_2) \dots \Phi(k, m_r).$$

is a multivariate multiplicative function, $m = \text{lcm}(m_1, \dots, m_r)$.

Liskovets (Integers, 2010) derived its multiplicative form,

It has many interpretations and applications, see Liskovets (Integers, 2010) and L. Toth (2011)

Number of toroidal ormaps

The number of oriented unrooted toroidal maps with e edges is

$$\frac{1}{2e} \left(\alpha(e) + \sum_{\ell|e} \phi_2(\ell) \mathcal{N}_1(e/\ell) \right),$$

where

$$\alpha(e) = \nu_{[0;2^4]}(e), \quad \text{if } e \equiv \pm 1, \pm 5 \pmod{12},$$

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;2,4^2]}(e/2), \quad \text{if } e \equiv \pm 2, \pm 4 \pmod{12},$$

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,3,6]}(e/3), \quad \text{if } e \equiv \pm 3 \pmod{12},$$

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,4^2]}(e/2) + 2\nu_{[0;2,3,6]}(e/3), \\ \text{if } e \equiv 0, 6 \pmod{12}.$$

Summary: Transfer from rooted to unrooted case

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2. Problem A of enumeration of rooted maps on admissible cyclic orbifolds, it is **purely combinatorial**.
3. Problem B of counting of the number of order preserving epimorphisms from fundamental groups of the above cyclic orbifolds onto cyclic groups, can be solved using techniques of **analytical number theory**.

Asymtotic number of maps of genus g

Bender and Canfield in 1986 proved that the number of maps of genus g with n edges is

$$m_g(n) \sim t_g n^{5(g-1)/2} 12^n,$$

where t_g is a constant computable via non-linear recursions.

The first two values are $t_0 = \frac{2}{\sqrt{\pi}}$ and $t_1 = \frac{1}{24}$.

Asymptotic behaviour of maps: chiral versus reflexible

Let n be a number of edges of a map on orientable surface, genus is not fixed!!!

Let $U(n)$ denote the number of Isoclasses of **sensed** maps on orientable surfaces,

Let $A(n)$ denote the number of Isoclasses of **reflexible** maps on orientable surfaces,

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This problem appears in a survey paper by T. Bender on map enumerations.

Asymptotic analysis: Main result - a short version

M. Drmota and N. (2012)

$$\log A(n) \sim \frac{1}{2} \log U(n) \sim (n/2) \log n$$

For sufficiently large n the number of reflexible maps takes approximately the square root of the number of maps

Leading term (elementary approach):

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But no idea how to work out the same way $A(n)$!

Methods used in the proof

Asymptotic properties of quickly increasing sequences, **E. A. Bender**, J. London Math. Soc. 1975 and their generalizations

Explicit formulas for $U(n)$ and $A(n)$ derived by **Breda, Mednykh, N.** DM 2010,

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