Complex Geometry and Toric Topology

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1. Moment-angle complexes and manifolds.

$\mathcal{K}$ an (abstract) simplicial complex on the set $[m] = \{1, \ldots, m\}$.

$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex. Always assume $\varnothing \in \mathcal{K}$.
Allow $\{i\} \notin \mathcal{K}$ for some i (ghost vertices).

Consider the unit polydisc in $\mathbb{C}^m$,

$$\mathbb{D}^m = \Bigl\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| \leqslant 1, \quad i = 1, \ldots, m\Bigr\}.$$

Given $I \subset [m]$, set

$$B_I := \Bigl\{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon |z_j| = 1 \text{ for } j \notin I\Bigr\}.$$

Define the moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m$$

It is invariant under the coordinatewise action of the standard torus

$$\mathbb{T}^m = \Bigl\{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon |z_i| = 1, \quad i = 1, \ldots, m\Bigr\}$$

on $\mathbb{C}^m$.

Constr 1 (polyhedral product)**.** Given spaces $W \subset X$ and $I \subset [m]$, set

$$(X,W)^I = \bigl\{(x_1,\dots,x_m) \in X^m \colon x_j \in W \text{ for } j \notin I\bigr\} \cong \prod_{i\in I} X \times \prod_{i\notin I} W,$$

and define the polyhedral product of (X,W) as

$$(X,W)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} (X,W)^I \subset X^m.$$

Then $\mathcal{Z}_{\mathcal{K}} = (\mathbb{D},\mathbb{T})^{\mathcal{K}}$, where $\mathbb{T}$ is the unit circle.

Another example is the complement of a coordinate subspace arrangement:

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\dots,i_k\}\notin\mathcal{K}} \{z \in \mathbb{C}^m \colon z_{i_1} = \ldots = z_{i_k} = 0\},$$

namely,

$$U(\mathcal{K}) = (\mathbb{C},\mathbb{C}^\times)^{\mathcal{K}} = \bigcup_{I\in\mathcal{K}} \Bigl(\prod_{i\in I}\mathbb{C} \times \prod_{i\notin I}\mathbb{C}^\times\Bigr),$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Clearly, $\mathcal{Z}_{\mathcal{K}} \subset U(\mathcal{K})$. Moreover, $\mathcal{Z}_{\mathcal{K}}$ is a $\mathbb{T}^m$-equivariant deformation retract of $U(\mathcal{K})$ for every $\mathcal{K}$ [**Buchstaber-P**].

Prop 1. *Assume* $|\mathcal{K}| \cong S^{n-1}$ *(a sphere triangulation with* m *vertices). Then* $\mathcal{Z}_{\mathcal{K}}$ *is a closed manifold of dimension* $m+n$*.*

We refer to such $\mathcal{Z}_{\mathcal{K}}$ as moment-angle manifolds.

If $\mathcal{K} = \mathcal{K}_P$ is the dual triangulation of a simple convex polytope P, then $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}_P}$ embeds in $\mathbb{C}^m$ as a nondegenerate (transverse) intersection of $m-n$ real quadratic hypersurfaces. Therefore, $\mathcal{Z}_P$ can be smoothed canonically.

Now assume $\mathcal{K}$ is the underlying complex of a complete simplicial fan Σ (a starshaped sphere).

A fan is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex cones in $\mathbb{R}^n$ such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each.

A fan $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ is complete if $\sigma_1 \cup \dots \cup \sigma_s = \mathbb{R}^n$.

Let Σ be a simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones generated by $\mathbf{a}_1, \dots, \mathbf{a}_m$. Its underlying simplicial complex is

$$\mathcal{K}_\Sigma = \Big\{ I \subset [m] \colon \{\mathbf{a}_i \colon i \in I\} \text{ spans a cone of } \Sigma \Big\}$$

Note: Σ is complete iff $|\mathcal{K}_\Sigma|$ is a triangulation of S^{n-1}.

Given Σ with 1-cones generated by $\mathbf{a}_1, \ldots, \mathbf{a}_m$, define a map

$$A\colon \mathbb{R}^m \to \mathbb{R}^n, \quad \mathbf{e}_i \mapsto \mathbf{a}_i,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the standard basis of $\mathbb{R}^m$. Set

$$\mathbb{R}^m_> = \{(y_1, \ldots, y_m) \in \mathbb{R}^m \colon y_i > 0\},$$

and define

$$R := \exp(\operatorname{Ker} A) = \Big\{(y_1, \ldots, y_m) \in \mathbb{R}^m_> \colon \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u}\rangle} = 1 \text{ for all } \mathbf{u} \in \mathbb{R}^n\Big\},$$

$R \subset \mathbb{R}^m_>$ acts on $U(\mathcal{K}_\Sigma) \subset \mathbb{C}^m$ by coordinatewise multiplications.

Thm 1. *Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with m one-dimensional cones, and let $\mathcal{K} = \mathcal{K}_\Sigma$ be its underlying simplicial complex. Then*

(a) the group $R \cong \mathbb{R}^{m-n}$ acts on $U(\mathcal{K})$ freely and properly, so the quotient $U(\mathcal{K})/R$ is a smooth $(m+n)$-dimensional manifold;

(b) $U(\mathcal{K})/R$ is $\mathbb{T}^m$-equivariantly homeomorphic to $\mathcal{Z}_\mathcal{K}$.

Therefore, $\mathcal{Z}_\mathcal{K}$ can be smoothed canonically.

2. Complex-analytic structures.

We shall show that the even-dimensional moment-angle manifold $\mathcal{Z}_\mathcal{K}$ corresponding to a complete simplicial fan admits a structure of a complex manifold. The idea is to replace the action of $\mathbb{R}^{m-n}_>$ on $U(\mathcal{K})$ (whose quotient is $\mathcal{Z}_\mathcal{K}$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

Rem 1. Complex structures on *polytopal* moment-angle manifolds $\mathcal{Z}_P$ were described by Bosio and Meersseman. They identified $\mathcal{Z}_P$ with a class of complex manifolds known as LVM-manifolds (named after López de Medrano, Verjovsky and Meersseman).

Assume $m-n$ is even from now on. We can always achieve this by formally adding an 'empty' one-dimensional cone to Σ; this corresponds to adding a ghost vertex to $\mathcal{K}$, or multiplying $\mathcal{Z}_\mathcal{K}$ by a circle.

Set $\ell = \frac{m-n}{2}$.

Constr 2. Choose a linear map $\Psi\colon \mathbb{C}^\ell \to \mathbb{C}^m$ satisfying the two conditions:

(a) $\mathrm{Re}\circ\Psi\colon \mathbb{C}^\ell \to \mathbb{R}^m$ is a monomorphism.

(b) $A\circ\mathrm{Re}\circ\Psi = 0$.

The composite map of the top line in the following diagram is zero:

$$\begin{array}{ccccccc}
\mathbb{C}^\ell & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\mathrm{Re}} & \mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\
 & & \big\downarrow \exp & & \big\downarrow \exp & & \big\downarrow \exp \\
 & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}^m_> & \xrightarrow{\exp A} & \mathbb{R}^n_>
\end{array}$$

where $|\cdot|$ denotes the map $(z_1,\ldots,z_m)\mapsto(|z_1|,\ldots,|z_m|)$. Now set

$$C = \exp\Psi(\mathbb{C}^\ell) = \Big\{\Big(e^{\langle\psi_1,\mathbf{w}\rangle},\ldots,e^{\langle\psi_m,\mathbf{w}\rangle}\Big)\in(\mathbb{C}^\times)^m\Big\}$$

where $\mathbf{w}=(w_1,\ldots,w_\ell)\in\mathbb{C}^\ell$, ψ_i denotes the ith row of the $m\times\ell$-matrix $\Psi=(\psi_{ij})$.

Then $C\cong\mathbb{C}^\ell$ is a complex-analytic (but not algebraic) subgroup in $(\mathbb{C}^\times)^m$. It acts on $U(\mathcal{K})$ by holomorphic transformations.

Ex 1. Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $n=0$, $m=2$, $\ell=1$, and $A\colon \mathbb{R}^2 \to 0$ is a zero map. Let $\Psi\colon \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$C = \left\{(e^z, e^{\alpha z})\right\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) of Constr 2 is void, while (a) is equivalent to that $\alpha \notin \mathbb{R}$. Then $\exp\Psi\colon \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/C$ with the natural complex structure is a complex torus $T^2_{\mathbb{C}}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/C \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T^2_{\mathbb{C}}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on 2ℓ elements (so that $n=0$, $m=2\ell$), we may obtain any complex torus $T^{2\ell}_{\mathbb{C}}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/C$.

Thm 2. *Let* Σ *be a complete simplicial fan in* $\mathbb{R}^n$ *with* m *one-dimensional cones, and let* $\mathcal{K} = \mathcal{K}_\Sigma$ *be its underlying simplicial complex. Assume that* $m - n = 2\ell$. *Then*

(a) *the holomorphic action of the group* $C \cong \mathbb{C}^\ell$ *on* $U(\mathcal{K})$ *is free and proper, so the quotient* $U(\mathcal{K})/C$ *is a compact complex* $(m-\ell)$*-manifold;*

(b) *there is a* $\mathbb{T}^m$*-equivariant diffeomorphism* $U(\mathcal{K})/C \cong \mathcal{Z}_\mathcal{K}$ *defining a complex structure on* $\mathcal{Z}_\mathcal{K}$ *in which* $\mathbb{T}^m$ *acts holomorphically.*

Ex 2 (Hopf manifold)**.** Let Σ be the complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n+1$ vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_1 - \ldots - \mathbf{e}_n$.

To make $m-n$ even we add one 'empty' 1-cone. We have $m = n+2$, $\ell = 1$. Then $A\colon \mathbb{R}^{n+2} \to \mathbb{R}^n$ is given by the matrix $(\mathbf{0}\; I\; -\mathbf{1})$, where I is the unit $n \times n$ matrix, and $\mathbf{0}$, $\mathbf{1}$ are the n-columns of zeros and units respectively.

We have that $\mathcal{K}$ is the boundary of an n-dim simplex with $n+1$ vertices and 1 ghost vertex, $\mathcal{Z}_\mathcal{K} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\Psi\colon \mathbb{C} \to \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \ldots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{R}$. Then

$$C = \left\{ (e^z, e^{\alpha z}, \ldots, e^{\alpha z})\colon z \in \mathbb{C} \right\} \subset (\mathbb{C}^\times)^{n+2},$$

and $\mathcal{Z}_\mathcal{K}$ acquires a complex structure as the quotient $U(\mathcal{K})/C$:

$$\mathbb{C}^\times \times \left(\mathbb{C}^{n+1} \setminus \{0\}\right) \Big/ \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z} \mathbf{w})\} \cong \left(\mathbb{C}^{n+1} \setminus \{0\}\right) \Big/ \{\mathbf{w} \sim e^{2\pi i \alpha} \mathbf{w}\},$$

where $t \in \mathbb{C}^\times$, $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. The latter quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ is known as the Hopf manifold.

3. Holomorphic bundles over toric varieties.

Manifolds $\mathcal{Z}_{\mathcal{K}}$ corresponding to complete *regular* simplicial fans are total spaces of holomorphic principal bundles over toric varieties with fibre a complex torus. This allows us to calculate invariants of the complex structures on $\mathcal{Z}_{\mathcal{K}}$.

A toric variety is a normal algebraic variety X on which an algebraic torus $(\mathbb{C}^{\times})^n$ acts with a dense (Zariski open) orbit.

Toric varieties are classified by rational fans. Under this correspondence,

$$\begin{aligned}
\text{complete fans} &\longleftrightarrow \text{compact varieties}\\
\text{normal fans of polytopes} &\longleftrightarrow \text{projective varieties}\\
\text{regular fans} &\longleftrightarrow \text{nonsingular varieties}\\
\text{simplicial fans} &\longleftrightarrow \text{orbifolds}
\end{aligned}$$

Σ complete, simplicial, rational;
$\mathbf{a}_1, \ldots, \mathbf{a}_m$ primitive integral generators of 1-cones;
$\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n$.

Constr 3 ('Cox construction'). Let $A_{\mathbb{C}}\colon \mathbb{C}^m \to \mathbb{C}^n$, $\mathbf{e}_i \mapsto \mathbf{a}_i$,

$$\exp A_{\mathbb{C}}\colon (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n,$$
$$(z_1, \ldots, z_m) \mapsto \Bigl(\prod_{i=1}^m z_i^{a_{i1}}, \ldots, \prod_{i=1}^m z_i^{a_{in}} \Bigr)$$

Set $G = \operatorname{Ker} \exp A_{\mathbb{C}}$.
This is an $(m-n)$-dimensional algebraic subgroup in $(\mathbb{C}^\times)^m$.
It acts almost freely (with finite isotropy subgroups) on $U(\mathcal{K}_\Sigma)$.
If Σ is regular, then $G \cong (\mathbb{C}^\times)^{m-n}$ and the action is free.

$V_\Sigma = U(\mathcal{K}_\Sigma)/G$ the toric variety associated to Σ.
The quotient torus $(\mathbb{C}^\times)^m/G \cong (\mathbb{C}^\times)^n$ acts on V_Σ with a dense orbit.

Observe that $\mathbb{C}^\ell \cong C \subset G_\Sigma \cong (\mathbb{C}^\times)^m$ as a complex subgroup.

Prop 2.

(a) The toric variety V_Σ is homeomorphic to the quotient of $\mathcal{Z}_{\mathcal{K}_\Sigma}$ by the holomorphic action of G/C.

(b) If Σ is regular, then there is a holomorphic principal bundle $\mathcal{Z}_{\mathcal{K}_\Sigma} \to V_\Sigma$ with fibre the compact complex torus G/C of dimension ℓ.

Rem 2. For singular varieties V_Σ the quotient projection $\mathcal{Z}_{\mathcal{K}_\Sigma} \to V_\Sigma$ is a holomorphic principal Seifert bundle for an appropriate orbifold structure on V_Σ.

4. Submanifolds and analytic subsets.

The complex structure on $\mathcal{Z}_{\mathcal{K}}$ is determined by two pieces of data:
– the complete simplicial fan Σ with generators $\mathbf{a}_1, \ldots, \mathbf{a}_m$;
– the ℓ-dimensional holomorphic subgroup $C \subset (\mathbb{C}^\times)^m$.

If this data is *generic* (in particular, the fan Σ is not rational), then there is no holomorphic principal torus fibration $\mathcal{Z}_{\mathcal{K}} \to V_\Sigma$ over a toric variety V_Σ.

However, there still exists a holomorphic ℓ-dimensional *foliation* $\mathcal{F}$ with a transverse Kähler form $\omega_{\mathcal{F}}$. This form can be used to describe submanifolds and analytic subsets in $\mathcal{Z}_{\mathcal{K}}$.

Consider the complexified map $A_{\mathbb{C}} \colon \mathbb{C}^m \to \mathbb{C}^n$, $\mathbf{e}_i \mapsto \mathbf{a}_i$. and the following complex $(m-n)$-dimensional subgroup in $(\mathbb{C}^\times)^m$:

$$G = \exp(\operatorname{Ker} A_{\mathbb{C}}) = \Bigl\{\Bigl(e^{z_1}, \ldots, e^{z_m}\Bigr) \in (\mathbb{C}^\times)^m \colon (z_1, \ldots, z_m) \in \operatorname{Ker} A_{\mathbb{C}}\Bigr\}.$$

Note $C \subset G$.

The group G acts on $U(\mathcal{K})$, and its orbits define a holomorphic foliation on $U(\mathcal{K})$. Since $G \subset (\mathbb{C}^\times)^m$, this action is free on open subset $(\mathbb{C}^\times)^m \subset U(\mathcal{K})$, so that the generic leaf of the foliation has complex dimension $m - n = 2\ell$.

The ℓ-dimensional closed subgroup $C \subset G$ acts on $U(\mathcal{K})$ freely and properly by Theorem 2, so that $U(\mathcal{K})/C$ carries a holomorphic action of the quotient group $D = G/C$.

$\mathcal{F}$: the holomorphic foliation on $U(\mathcal{K})/C \cong \mathcal{Z}_{\mathcal{K}}$ by the orbits of D.

The subgroup $G \subset (\mathbb{C}^\times)^m$ is closed if and only if it is isomorphic to $(\mathbb{C}^\times)^{2\ell}$; in this case the subspace $\operatorname{Ker} A \subset \mathbb{R}^m$ is rational. Then Σ is a rational fan and V_Σ is the quotient $U(\mathcal{K})/G$. The foliation $\mathcal{F}$ gives rise to a holomorphic principal Seifert fibration $\pi\colon \mathcal{Z}_\mathcal{K} \to V_\Sigma$ with fibres compact complex tori G/C.

For a generic configuration of nonzero vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$, G is biholomorphic to $\mathbb{C}^{2\ell}$ and $D = G/C$ is biholomorphic to $\mathbb{C}^\ell$.

A $(1,1)$-form $\omega_{\mathcal{F}}$ on the complex manifold $\mathcal{Z}_{\mathcal{K}}$ is called transverse Kähler with respect to the foliation $\mathcal{F}$ if
(a) $\omega_{\mathcal{F}}$ is closed, i.e. $d\omega_{\mathcal{F}} = 0$;
(b) $\omega_{\mathcal{F}}$ is nonnegative and the zero space of $\omega_{\mathcal{F}}$ is the tangent space of $\mathcal{F}$.

A complete simplicial fan Σ in $\mathbb{R}^n$ is called weakly normal if there exists a (not necessarily simple) n-dimensional polytope P such that Σ is a simplicial subdivision of the normal fan Σ_P.

Thm 3. *Assume that Σ is a weakly normal fan. Then there exists an exact $(1,1)$-form $\omega_{\mathcal{F}}$ on $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/C$ which is transverse Kähler for the foliation $\mathcal{F}$ on the dense open subset $(\mathbb{C}^\times)^m/C \subset U(\mathcal{K})/C$.*

For each $J \subset [m]$, define the corresponding coordinate submanifold in $\mathcal{Z}_\mathcal{K}$ by

$$\mathcal{Z}_{\mathcal{K}_J} = \{(z_1, \ldots, z_m) \in \mathcal{Z}_\mathcal{K} \colon z_i = 0 \quad \text{for } i \notin J\}.$$

Obviously, $\mathcal{Z}_{\mathcal{K}_J}$ is identified with the quotient of

$$U(\mathcal{K}_J) = \{(z_1, \ldots, z_m) \in U(\mathcal{K}) \colon z_i = 0 \quad \text{for } i \notin J\}$$

by $C \cong \mathbb{C}^\ell$. In particular, $U(\mathcal{K}_J)/C$ is a complex submanifold in $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/C$.

Observe that the closure of any $(\mathbb{C}^\times)^m$-orbit of $U(\mathcal{K})$ has the form $U(\mathcal{K}_J)$ for some $J \subset [m]$ (in particular, the dense orbit corresponds to $J = [m]$). Similarly, the closure of any $(\mathbb{C}^\times)^m/C$-orbit of $\mathcal{Z}_\mathcal{K} \cong U(\mathcal{K})/C$ has the form $\mathcal{Z}_{\mathcal{K}_J}$.

Thm 4. *Assume that the data defining a complex structure on* $\mathcal{Z}_{\mathcal{K}} = U(\mathcal{K})/C$ *is generic. Then any divisor of* $\mathcal{Z}_{\mathcal{K}}$ *is a union of coordinate divisors.*

Furthermore, if Σ *is a weakly normal fan, then any compact irreducible analytic subset* $Y \subset \mathcal{Z}_{\mathcal{K}}$ *of positive dimension is a coordinate submanifold.*

Cor 1. *Under generic assumptions, there are no non-constant meromorphic functions on* $\mathcal{Z}_{\mathcal{K}}$*.*

[BP] Victor Buchstaber and Taras Panov. *Torus Actions and Their Applications in Topology and Combinatorics.* University Lecture Series, vol. **24**, Amer. Math. Soc., Providence, R.I., 2002.

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