On an approach to the calculation of volumes in spaces of constant curvature

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INTRODUCTION

It is well known that the volume $\mathbf{V}$ of a body $\mathbf{D}$ in the three-dimensional Euclidean space can be calculated by the following surface integral

$$\iint_{\partial D} x dy \wedge dz + y dz \wedge dx + z dx \wedge dy = 3V, \qquad (1)$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are components of the vector-position of points on the boundary $\partial\mathbf{D}$ of the body $\mathbf{D}$ and $\mathbf{V}$ is the oriented volume of $\mathbf{D}$. If we start from a compact embedded surface $\mathbf{S}$ then the above integral over $\mathbf{S}$ gives the volume enclosed by the surface $\mathbf{S}$. If $\mathbf{S}$ is a compact surface with some self-intersections then the value of integral defines so-called algebraic volume enclosed by $\mathbf{S}$.

Our purpose is to present for volumes in spaces of constant curvature of any dimension a formula analogue to the formula (1) and to give some its applications.

CONSIDERED METRICS

We will consider the metrics of n-spaces of constant curvature in the following form

$$ds^2 = \frac{1}{(1+ar^2)^2}(dx_1^2 + ... + dx_n^2), \qquad (2)$$

where $\mathbf{r^2 = x_1^2 + ... + x_n^2}$ **and the coefficient** $\mathbf{a}$ **is a constant. If** $\mathbf{a = 0}$ **we have the Euclidean case; the values** $\mathbf{a > 0}$ **correspond to the spherical metric with curvature** $\mathbf{K = 4a}$**; when** $\mathbf{a < 0}$ **we have a metric of Lobachevsky n-space with curvature** $\mathbf{K = 4a}$**. The domain of existence of the metric (2) we note as** $\mathbf{B}$**, and in the Euclidean case we have** $\mathbf{B : 0 \leq r < \infty}$**, for a spherical metric** $\mathbf{B : 0 \leq r \leq \infty}$**; finally in the hyperbolic case** $\mathbf{B : 0 \leq r < \frac{1}{\sqrt{-a}}}$**.**

An other class of metrics is defined for hyperbolic metrics only. They are related to Poincaré upper half-space presentation of Lobachevsky space with curvature $\mathbf{K < 0}$**:**

$$\mathbf{ds^2 = \frac{dx_1^2 + ... + dx_n^2}{(-K)x_n^2}, x_n > 0.} \qquad \textbf{(2a)}$$

THE MAIN THEOREMS

It is easy to prove the following lemma

Lemma 1. *Let* $\mathbf{D} \subset \mathbf{B}$ *is a compact body with piece-wise smooth boundary* $\partial\mathbf{D}$. *Then the following integral equality holds*

$$\mathbf{2(k-1)J_k = (2k-n-2)J_{k-1} + I_{k-1}}, \tag{3}$$

where

$$\mathbf{J_k = \int\limits_D \frac{dx_1 \wedge ... \wedge dx_n}{(1+ar^2)^k}, k \geq 1}$$

$$\mathbf{I_k = \int\limits_{\partial D} \sum_{i=1}^{n} \frac{(-1)^{i-1} x_i dx_1 \wedge ... \hat{d}x_i ... \wedge dx_n}{(1+ar^2)^k}, k \geq 0},$$

and the sign $\mathbf{\hat{d}x_i}$ *means that this factor in the product is omitted.*

Using the formula (3) in the case of an even $\mathbf{n = 2m}$ **we obtain**

$$\mathbf{J_n = \sum_{k=0}^{m-1} \frac{(2m-2)!!(m+k-1)!}{2^{m-k}(2k)!!(2m-1)!} I_{m+k}}$$

that is the integral over the body $\mathbf{D}$ **is expressed by integrals over the boundary of** $\mathbf{D}$**.**

In the case of an odd $n = 2m + 1$ we have

$$J_n = \frac{P(2-n, n-1)}{(n-1)!2^{n-1}} J_1 + \sum_{k=1}^{n-1} \frac{P(2-n, n-1)(k-1)!}{P(2-n, k)(n-1)!2^{n-k}} I_k,$$

where $P(2-n, k)$ is the product of k succesively increasing odd numbers beginning from $2-n$. The integral J_1 can be expressed by some integral over the surface too. It turns out that in reality our both considerations for even and odd cases of n can be unified together.

Lemma 2. *The following integral equality holds*

$$J_k = \int_{\partial D} \sum_{i=1}^{n} \left(\frac{(-1)^{i-1} x_i F_k(r)}{r^n} \right) dx_1 \wedge ... \hat{d}x_i ... \wedge dx_n, \forall k \tag{4}$$

where $r^2 = x_1^2 + ... + x_n^2$ *and*

$$F_k(r) = \int_0^r \frac{t^{n-1}}{(1+at^2)^k} dt \tag{5}$$

Now we remark that for the metric form (2) the integral $\mathbf{J_n = \int\limits_D \sqrt{det(g_{ij})} dx_1 \wedge ... \wedge dx_n}$ **gives the value of volume of the body** D **and using the lemma 2 we arrive to the theorem**

Theorem 1. *The volume* $\mathbf{V}$ *of a body* $\mathbf{D}$ *calculated for the metric (2) can be found by the following surface integral*

$$\mathbf{V(D) = \int\limits_{\partial D} \sum_{i=1}^{n} \left(\frac{(-1)^{i-1} x_i F_n(r)}{r^n} \right) dx_1 \wedge ... \hat{d}x_i ... \wedge dx_n,} \tag{6}$$

where the function $F_n(r)$ *is defined by the expression (5).*

If we introduce the function

$$\Phi(r) = \int_0^r t^{1-n} F_n(t) dt$$

then the integral (6) can be interpreted as the flow (in Euclidean sense) of the vector field $\mathrm{grad}\Phi$ across the hypersurface ∂D. For the Euclidean space (the case $a = 0$) we have $\Phi = \frac{t^2}{2n}$ and the integral (6) presents the $1/n$ part of the flow of vector-position across the surface, just in accordance with the known fact.

If some compact hypersurface S has self-intersections then the integral (6) over it can be considered as a definition of the algebraic volume enclosed by this surface.

The formulae for the hyperbolic metrics in the upper half-space presentation are more nice and effective.

Theorem 2. *Let D be a compact body with a piece-wise boundary situated in the upper half-space $x_n > 0$. Then its hyperbolic volume V can be calculated as follows*

$$V = \frac{1}{(-K)^{n/2}} \int\limits_D \frac{dx_1 \wedge ... \wedge dx_n}{x_n^n} =$$

$$\frac{1}{(n-1)(-K)^{n/2}} \int\limits_{\partial D} \frac{(-1)^n dx_1 \wedge ... \wedge dx_{n-1}}{x_n^{n-1}} \qquad (7)$$

Remark 1. *If the surface* $\mathbf{S} = \partial \mathbf{D}$ *can be divided in two parts each of which is presented by equations* $\mathbf{x_n = f_1(x_1, ..., x_{n-1})}$, $\mathbf{x_n = f_2(x_1, ..., x_{n-1})}$ *over some domain* $\mathbf{\Omega} \subset \mathbb{R}^{\mathbf{n-1}}$ *then the volume enclosed by the surface* S *is given by the integral*

$$\mathbf{V = \frac{(-1)^n}{(n-1)(-K)^{n/2}}[\int\limits_{\Omega} \frac{dx_1 \wedge ... \wedge dx_{n-1}}{f_1^{n-1}(x_1, ..., x_{n-1})} - \int\limits_{\Omega} \frac{dx_1 \wedge ... \wedge dx_{n-1}}{f_2^{n-1}(x_1, ..., x_{n-1})}]}$$

For example using this formula one can calculate the volume of a cylinder-like body

$$\mathbf{B : x^2 + y^2 \le r^2, \sqrt{R_1^2 - (x^2 + y^2)} \le z \le \sqrt{R_2^2 - (x^2 + y^2)},}$$
$$\mathbf{r < R_1 < R_2}$$

in three-dimensional Lobachevsky space with the curvature $\mathbf{K}$**. We obtain**

$$\mathbf{V_B = \frac{\pi}{2(-K)^{\frac{3}{2}}} \ln \frac{R_1^2(R_2^2 - r^2)}{R_2^2(R_1^2 - r^2)}.}$$

Remark 2. *One can propose other forms of integral presentation too:*

$$\mathbf{V = \frac{1}{2(n-1)(-K)^{n/2}}[\int\limits_{\partial D} \sum_{j=1}^{n-1} \frac{(-1)^{j-1} x_j}{x_n^n} dx_1 \wedge ... \hat{d}x_j ... \wedge dx_n +}$$
$$\mathbf{\int\limits_{\partial D} \frac{(-1)^n dx_1 \wedge ... \wedge dx_{n-1}}{x_n^{n-1}}].}$$

In general one can take any combination of terms of view $\mathbf{a_j \frac{(x_j - b_j) dx_1 \wedge ... \hat{d}x_j ... dx_n}{x_n^n}, j \le n-1}$, *and* $\mathbf{a_n \frac{dx_1 \wedge ... \wedge dx_{n-1}}{x_n^{n-1}}}$ *with some constant coefficients* $\mathbf{a_j, b_j}$ *under the condition that the application of Stokes formule this combination will give as a result the first integral in (7).*

SOME APPLICATIONS

Now we want to propose some applications of above formulae. The first applications consists in an evident remark that we can easily calculate the volumes of bodies which are bounded by sufficiently simple surfaces in the Euclidean sense. For example if we consider a body bounded in the dimension $n = 3$ for the metric (2) by the sphere $S : x_1^2 + x_2^2 + x_3^2 = R^2$ then the volume V of this body will be given by the formula

$$V = \int\limits_S \frac{F_3(r)}{r^3}(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2),$$

where the function $F_3(r)$ is defined by the expression

$$F_3(r) = \int_0^r \frac{t^2}{(1 + at^2)^3} dt.$$

On the sphere S we have $r = R$ and in the hyperbolic case $a < 0$ we obtain

$$V = 4\pi \left(\frac{1}{8(1+q)} - \frac{1}{4(1+q)^2} + \frac{1}{16\sqrt{-q}} \ln \frac{1+\sqrt{-q}}{1-\sqrt{-q}} \right) \frac{R^3}{q}, \;\; q = aR^2 < 0$$

with the asymptotic behavior

$$V = (\frac{4}{3}\pi - \frac{3}{5}\pi q + ...)R^3$$

when $a \to 0$ that is when the hyperbolic metric (2) tends to the Euclidean one.

In the spherical case $\mathbf{a > 0}$ we have

$$\mathbf{V = 4\pi \left(\frac{1}{8(1+q)} - \frac{1}{4(1+q)^2} + \frac{1}{8\sqrt{q}} arctg(\sqrt{q}) \right) \frac{R^3}{q}, \; q = aR^2 > 0}$$

with the asymptotic behavior

$$\mathbf{V = (\frac{4}{3}\pi - \frac{3}{5}\pi q + ...)R^3}$$

when $\mathbf{a \to 0}$ that is when the spherical metric (2) tends to the Euclidean one.

Of course more interesting applications are for the calculations of volumes of the bodies really having some geometrical sense in hyperbolic or spherical spaces.

Algebraic area of a polygon on the hyperbolic plane

Let $\mathbf{P}$ be a $\mathbf{n}$-gon on the Lobachevsky plane presented by the Poincaré model with the metric

$$\mathbf{ds^2 = \frac{dx^2 + dy^2}{(-K)y^2},\ \ y > 0}.$$

The theorem 2 gives us a formula for the area of a domain via an integral along its boundary. This integral over any closed curve $\mathbf{L}$ (may be with some self-intersections) can be considered as the definition of algebraic area enclosed by the curve $\mathbf{L}$. So the algebraic area $\mathbf{S_L}$ is defined as follows

$$\mathbf{S_L = \frac{1}{-K} \oint\limits_{L} \frac{dx}{y}}. \qquad (8)$$

In the case of an oriented $\mathbf{n}$-gon $\mathbf{P}$ with vertices $\mathbf{M_1, ..., M_n}$ the integral in (8) is a sum of integrals along each side $\mathbf{M_iM_{i+1}, 1 \leq i \leq n}$, where $\mathbf{M_{n+1} = M_1}$. Let $\mathbf{(x_i, y_i)}$ be coordinates of the vertex $\mathbf{M_i}$. The side $\mathbf{M_iM_{i+1}}$ is situated on the semi-circle $\mathbf{S_i}$ with an equation $\mathbf{(x - a_i)^2 + y^2 = R_i^2, y > 0}$, and the points of arc $\mathbf{M_iM_{i+1}}$ have coordinates

$$\mathbf{x = a_i + R_i \cos\varphi, y = R_i \sin\varphi}.$$

One has the equations

$$\mathbf{x_i = a_i + R_i \cos\varphi_i^+,\ \ y_i = R_i \sin\varphi_i^+,}$$
$$\mathbf{x_{i+1} = a_i + R_i \cos\varphi_{i+1}^-,\ \ y_{i+1} = R_i \sin\varphi_{i+1}^-,}$$

Then the integral (8) over the side $\mathbf{M_iM_{i+1}}$ is equal to the expression

$$-\int_{\mathbf{x_i}}^{\mathbf{x_{i+1}}} \mathbf{d}\varphi = \varphi_{\mathbf{i}}^+ - \varphi_{\mathbf{i+1}}^-.$$

Geometrically the difference $\varphi_{\mathbf{i}}^+ - \varphi_{\mathbf{i+1}}^-$ is equal to the angle $\mathbf{\Delta_i}\varphi$ between radii going from the center $\mathbf{(a_i, 0)}$ of the circle $\mathbf{S_i}$ to points $\mathbf{M_i}$ and $\mathbf{M_{i+1}}$ taken with a corresponding sign. Therefore for the area $\mathbf{S_P}$ we have the following formula

$$-\mathbf{KS_P} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{\Delta_i}\varphi.$$

But this formula is not very convenient because to calculate the terms of the sum we needed to introduce a special model of Lobachevsky plane. To obtain an invariant form for the considered area we present the terms of sum by groups of view

$$(\varphi^{+}_{i+1} - \varphi^{-}_{i+1}), 0 \leq i \leq n-1.$$

The radii $a_i M_{i+1}$ and $a_{i+1} M_{i+1}$ are normals at the point M_{i+1} to the arcs $M_i M_{i+1}$ and $M_{i+1} M_{i+2}$ correspondingly and so they are related with the tangent directions to these arcs and consequently with the angles between them. We define at any vertex M_i so called interior angle γ_i as the value of the angle at M_i remaining at left side when we go along the polygon following its orientation. We take always the value of γ as $0 < \gamma < 2\pi$ (we suppose that there are no cases $\gamma = 0$ or 2π). The difference $\varphi^{+}_{i+1} - \varphi^{-}_{i+1}$ is just the sign-valued angle α_{i+1} from the normal at M_{i+1} for the arc $M_i M_{i+1}$ to the normal at the same point M_{i+1} for the arc $M_{i+1} M_{i+2}$ (both of normals are taken as continuations of the corresponding radii to M_{i+1} from a_i and a_{i+1}).

It turns out that the interior angle γ_{i+1} at the vertex M_{i+1} is related with the angle α_{i+1} by the following equalities : 1) $\alpha_{i+1} = -\gamma_{i+1}$ or 2) $\alpha_{i+1} = \pi - \gamma_{i+1}$ or 3) $\alpha_{i+1} = 2\pi - \gamma_{i+1}$ in dependence of the positions of tangents to the arcs relatively the arcs themselves. For these positions there are 8 possibilities and all they are explicitly described geometrically.

So finally we have the following theorem

Theorem 3. *The algebraic area* S_P *enclosed by an oriented closed* n*-gon* P *is given by the formula*

$$-KS_P = m\pi - \sum_{i=1}^{n} \gamma_i, \qquad (9)$$

where m *is a positive integer less than* $2n$ *and* $0 < \gamma_i < 2\pi$ *are interior angles at vertices* $M_i, 1 \leq i \leq n$.

As to the exact value of m in the formula (9) for this we need to know how many times we have for the relation between α_i and γ_i the cases 1), 2) and 3. If the case 1) occurs m_1 times, the case 2) - m_2 times and the case 3) m_3 times then $m = m_2 + 2m_3$.

For example, for a triangle one can show that there are cases when $\mathbf{m_1 = 2, m_2 = 1, m_3 = 0}$ so its area is equal

$$\frac{\mathbf{1}}{\mathbf{-K}}[\pi - (\gamma_{\mathbf{1}} + \gamma_{\mathbf{2}} + \gamma_{\mathbf{3}})]$$

just in accordance with the classical result.

Remark 3. *It would be interesting to find a relation between the number m in (9) and the index of the curve composed by circular arcs and presenting a considered hyperbolic polygon.*

Algebraic volume of a polyhedron in a hyperbolic space.

Let $\mathbf{P}$ be a polyhedron of any combinatorial structure in a $\mathbf{n}$-dimensional Lobachevsky space presented by the Poincaré model with the metric

$$\mathbf{ds^2 = \frac{dx_1^2 + ... + dx_n^2}{(-K)x_n^2},\ \ x_n > 0}.$$

The theorem 2 gives us a formula for the volume of a domain via an integral along its boundary. This integral over any closed hypersurface $\mathbf{S}$ (may be with some self-intersections) can be considered as the definition of algebraic volume enclosed by the surface $\mathbf{S}$. So the algebraic volume $\mathbf{V_S}$ is defined as follows

$$\mathbf{V_S = \frac{1}{(n-1)(-K)^{n/2}} \int\limits_S \frac{(-1)^n dx_1 \wedge ... \wedge dx_{n-1}}{x_n^{n-1}}} \qquad (10)$$

For a polyhedron $\mathbf{P}$ the integral is the sum of integrals over oriented hyperfaces ω_i of $\mathbf{P}$. Any hypersurface is given as a part of a semi-sphere:

$$\mathbf{S_i} : (\mathbf{x_1} - \mathbf{a_{i1}})^2 + \ldots + (\mathbf{x_{n-1}} - \mathbf{a_{i,n-1}})^2 + \mathbf{x_n^2} = \mathbf{R_i^2}$$

(if ω_i is a domain on a hyperlane orthogonal to the plane $x_n = 0$ then the integral over this hyperface is equal to 0).

Let $\Omega_i \subset \mathbb{R}^{n-1}$ be the projection of ω_i on the hyperplane $x_n = 0$. If the semi-sphere S_i is oriented by its exterior normal then

$$\int\limits_{\omega_i} \frac{\mathbf{dx_1} \wedge \ldots \wedge \mathbf{dx_{n-1}}}{\mathbf{x_n^{n-1}}} = \int\limits_{\Omega_i} \frac{\mathbf{dx_1} \wedge \ldots \wedge \mathbf{dx_{n-1}}}{(\mathbf{R_i^2} - \mathbf{r_i^2})^{(n-1)/2}} \qquad (11)$$

where $\mathbf{r_i^2} = (\mathbf{x_1} - \mathbf{a_{i1}})^2 + \ldots + (\mathbf{x_{n-1}} - \mathbf{a_{i,n-1}})^2$. In one's turn the integral over Ω_i can be reduced to an integral over its boundary:

$$\frac{1}{\mathbf{R_i^{n-1}}} \int\limits_{\Omega_i} \frac{\mathbf{dx_1} \wedge \ldots \wedge \mathbf{dx_{n-1}}}{[1 - (\frac{\mathbf{r_i}}{\mathbf{R_i}})^2]^{(n-1)/2}} =$$

$$\int\limits_{\partial\Omega_i} \sum_{j=1}^{n-1} \frac{(-1)^{j-1}(\mathbf{x_j} - \mathbf{a_{ij}})\mathbf{F}(\frac{\mathbf{r_i}}{\mathbf{R_i}})}{\mathbf{r_i^{n-1}}} \mathbf{dx_1} \wedge \ldots \hat{\mathbf{dx_j}} \ldots \wedge \mathbf{dx_{n-1}}, \quad (12)$$

where the function F is defined as follows

$$\mathbf{F}(\mathbf{t}) = \int_0^t \frac{\tau^{n-2}}{(1 - \tau^2)^{(n-1)/2}} d\tau, \ \mathbf{t} = \frac{\mathbf{r_i}}{\mathbf{R_i}}.$$

If the considered polyhedron is a pseudo-manifold then any integral in the right side of (12) has its unique analogue belonging to the boundary of a face F_k coinciding to F_i. These two integrals should be responsible for one of terms in the famous Schläfli formula.

Remark 4. *It is curious to observe that the integral*

$$\int_{\Omega_i} \frac{dx_1 \wedge ... \wedge dx_{n-1}}{(R_i^2 - r_i^2)^{(n-1)/2}}$$

at the right side in (11) can be considered as the volume of the $(n-1)$*- dimensional polyhedral domain* Ω_i *provided with the metric*

$$ds^2 = \frac{1}{(R_i^2 - r_i^2)}(dx_1^2 + ... + dx_{n-1}^2) \qquad (13)$$

determined in the ball with radius R_i *and center* $(a_{i1}, ..., a_{i,n-1})$.

The metric (13) by its view is seeming to a metric of constant negative curvature given in the ball B but it is not a metric of constant curvature nevertheless one should be some interesting relations between it and the original metric.

Remark 5. *A formula analogue to (12) can be proven for metrics of view (2) too.*

Algebraic volume of tetrahedra in a hyperbolic space.

Let's consider the case of tetrahedra and present a method for finding their volume in function of length of edges. We will say that a hyperbolic tetrahedron in an n-dimensional Poincaré model of Lobachevsky space is in the standard position if its vertices can be numbered as $A_0, A_1, A_2, ..., A_n$ by such a manner that they have the coordinates

$$A_0(0,0,...,0,1), A_1(0,0,...,0,q), A_2(x_{21},0,...,0,x_{2n})(x_{21}>0),$$
$$A_3(x_{31},x_{32},0,...0,x_{3n})(x_{32}>0),...,$$
$$A_k(x_{k1},...,x_{k,k-1},0,...,0,x_{kn})(x_{k,k-1}>0),...,$$
$$A_{n-1}(x_{n-1,1},...,x_{n-1,n-2},0,x_{n-1,n})(x_{n-1,n-2}>0),$$
$$A_n(x_{n1},...,x_{n,n-1},x_{nn})(x_{n,n-1}>0)$$

(to have the positive orientation of the considered tetrahedron we should put the vertex A_1 "below" A_0 in case of even n (so one should be $q<1$) and $q>1$ in the case of odd n). We note that any tetrahedron can be disposed in the standard position by some motions in the space.

Lemma 3. *For a tetrahedron in the standard position the coordinates of its vertices are determined uniquely and explicitly by some elementary functions in lengths of its edges.*

In the standard position all hyperfaces of tetraedron except two ones are situated on the hyperplanes with equations of view $a_1x_1 + ... + a_{n-1}x_{n-1} = a_n$ and therefore in (11) all integrals over such hyperfaces are equal to 0. There are two faces only, namely $\omega_0 : A_0A_2, ..., A_n$ and $\omega_1 : A_1A_2, ..., A_n$ which are situated

on the semi-spheres

$$S_0 : (x_1 - a_{10})^2 + ..., +(x_{n-1} - a_{n-1,0})^2 + x_n^2 = R_0^2,$$
$$S_1 : (x_1 - a_{11})^2 + ..., +(x_{n-1} - a_{n-1,1})^2 + x_n^2 = R_1^2$$

The faces ω_0 and ω_1 are $(n-1)$-dimensional tetrahedra and they have the same projection Ω on the plane $x_n = 0$ with n vertices

$$A'_0(0, ..., 0), A'_2(x_{21}, 0, ..., 0, 0), ..., A'_n(x_{n1}, ..., x_{n,n-1}, 0)$$

so Ω is a $(n-1)$-dimensional tetrahedron with $(n-2)$-dimensional boundary composed by $(n-2)$- dimensional tetrahedra too.

By lemma 3 if we know the lengths of edges we know the coordinates of vertices of the considered tetrahedron so we can find by elementary functions the centers of spheres $\mathbf{S_0}$ and $\mathbf{S_1}$ and their radii and finally we can find the equations of faces.

Thus we have an algorithm for calculation of volumes of tetrahedra.

Let's consider the cases of dimensions $\mathbf{n = 3}$ and $\mathbf{n = 4}$.

The case $\mathbf{n = 3}$. The vertices of a tetrahedron in standard position have coordinates

$$\mathbf{A_0(0,0,1), A_1(0,0,q), A_2(x_{21},0,x_{23}), A_3(x_{31},x_{32},x_{33}).}$$

The vertices of $\mathbf{\Omega}$ on the plane $\mathbf{x_3 = 0}$ have coordinates

$$\mathbf{O(0,0), A'_2(x_{21},0), A'_3(x_{31},x_{32}).}$$

For the sphere $\mathbf{S_0}$ the function

$$\mathbf{F(\tau) = -\frac{1}{2}\ln(1-\tau^2), \tau = \frac{r_0}{R_0}}$$

where $\mathbf{r_0^2 = (x_1 - a_{01})^2 + (x_2 - a_{02})^2}$ and $\mathbf{R_0}$ is expressed explicitly by some formula in function of coordinates of vertices $\mathbf{A_0, A_2, A_3}$. The triangle $\mathbf{\Omega}$ has the sides with the following parameterized equations

$$\mathbf{OA'_2: \ x_1 = x_{21}t, x_2 = 0, 0 \le t \le 1.}$$
$$\mathbf{A'_2A'_3: \ x_1 = x_{21} + t(x_{31} - x_{21}), x_2 = x_{32}t, 0 \le t \le 1.}$$
$$\mathbf{A'_3O: \ x_1 = x_{31} - x_{31}t, x_2 = x_{32} - x_{32}t, 0 \le t \le 1.}$$

On these sides we have consequently:

$$\mathbf{r_0^2 = (x_{21}t - a_{01})^2 + a_{02}^2,}$$
$$\mathbf{r_0^2 = (x_{21} - a_{01} + t(x_{31} - x_{21}))^2 + (x_{32}t - a_{02})^2,}$$
$$\mathbf{r_0^2 = (x_{31} - a_{01} - x_{31}t)^2 + (x_{32} - a_{02} - x_{32}t)^2,}$$

and $\mathbf{F}$ is given everywhere by the corresponding expression $\mathbf{F(\frac{r_0}{R_0}) = -\frac{1}{2}\ln(1 - (\frac{r_0}{R_0})^2)}$. Now we have to substitute these expressions in the formula (12) and calculate the integrals

$$\mathbf{\frac{a_{02}x_{21}}{2}\int\limits_{OA_2'}\frac{\ln(1-(\frac{r_0}{R_0})^2}{r_0^2}dt+}$$
$$\mathbf{\frac{1}{2}\int\limits_{A_2'A_3'}[(x_{32}t - a_{02})(x_{31} - x_{21}) - (x_{21} + t(x_{31} - x_{21}) - a_{01})x_{32}]}$$
$$\frac{(\ln(1-(\frac{r_0}{R_0})^2)}{r_0^2}dt + \frac{1}{2}\int\limits_{A_3'O}[(x_{32} - x_{32}t - a_{02})(x_{31} - x_{21}) -$$
$$\mathbf{(x_{21} + t(x_{31} - x_{21} - a_{01})x_{32}]\frac{\ln(1-(\frac{r_0}{R_0})^2}{r_0^2}dt}$$

For the second face presenting a part of the semi-sphere passing by the vertices $\mathbf{A_1A_2A_3}$ we have to do the analogical calculations and after to take a difference between the results in dependence of the orientation of the tetrahedron.

We recall that all values of $\mathbf{x}$ and $\mathbf{a}$ with subscripts are known if we know the lengths of edges.

The case $\mathbf{n = 4}$**. In this case the vertices of the tetrahedron** $\mathbf{\Omega}$ **on the plane** $\mathbf{x_4 = 0}$ **are points**

$$\mathbf{O(0,0,0), A_2'(x_{21},0,0), A_3'(x_{31},x_{32},0), A_4'(x_{41},x_{42},x_{43})}$$

and they compose 4 faces with equations

$$\mathbf{OA_2'A_3': \ z = 0; \ OA_2'A_4': \ x_{43}y - x_{42}z = 0}$$

$$\mathbf{OA_3'A_4': \ (x_{33}x_{42} - x_{43}x_{32})x + (x_{31}x_{43} - x_{33}x_{41})y + (x_{32}x_{41} - x_{31}x_{42})z = 0}$$

$$\mathbf{A_2'A_3'A_4': x_{32}x_{43}(x - x_{41}) + x_{43}(x_{21} - x_{31})(y - x_{41}) +} (x_{32}(x_{21} - x_{41}) - x_{42}(x_{21} - x_{31}))(z - x_{43}) = 0.$$

Now for the semi-sphere $\mathbf{S_0}$ **with radius** $\mathbf{R_0}$ **passing by the vertices** $\mathbf{A_0A_2A_3A_4}$ **the function**

$$\mathbf{F = \frac{r_0}{\sqrt{R_0^2 - r_0^2}} - \arcsin(\frac{r_0}{R_0})}$$

The integrals in the right side of (12) should be taken over the above presented triangle faces of $\mathbf{\Omega}$ **but any such integral can be reduced to an integral over the 1-dimensional edges of** $\mathbf{\Omega}$**. We wanted to prove that these integrals (or their sum) give us elementary functions in coordinates of vertices but we did not succeeded to do it.**

So from our considerations some open questions arise.

1) To find a proof of Schläfli formula using the presentation (12).

2) Is it true that for any dimension $\mathbf{n}$ the calculation of volumes of hyperbolic tetrahedra are reducing finally to the integrals over some 1-dimensional straight segments related with Ω or its consequent projections? For dimensions $\mathbf{n = 2, 3, 4}$ it is true.

3) Is it true that for any even dimensions $\mathbf{n = 2k}$ the volume of a tetrahedron is presented by an elementary function in lengths of its edges? For $n = 2$ it is true.

4) Is it interesting and useful to compose a computer programm for numerical calculation of volumes of tetrahedra in low dimensions?