Simple loops on bridge spheres and Heegaard surfaces

– Subgroups of mapping class groups related to Heegaard splittings and bridge decompositions –

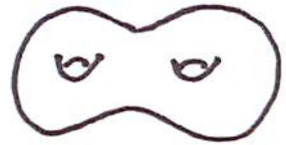
Dedicated to Professor Alexander Mednykh on the occasion of his 60th birthday

作間 誠 (広島大学)

Joint work with 李 東姫 (釜山大学)

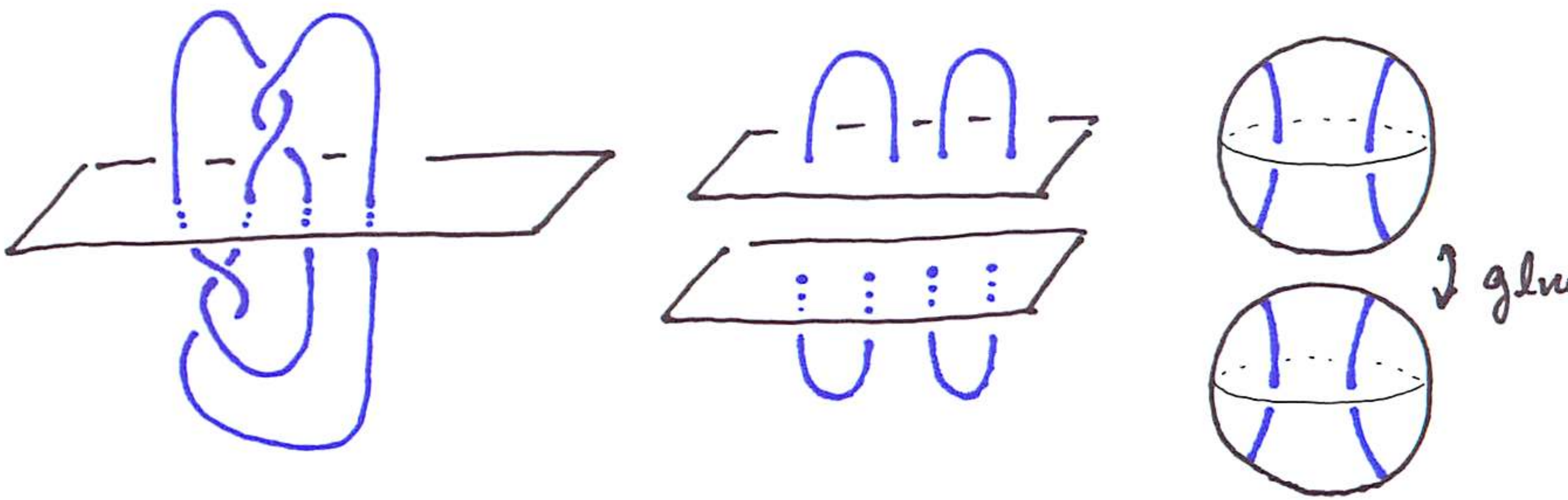
Joint work with 大鹿健一 (大阪大学)

Heegaard decomposition of a closed orientable 3-manifold

$M = V_1 \underset{S}{\cup} V_2$, where V_i =  : handlebody

$S = V_1 \cap V_2$: Heegaard surface

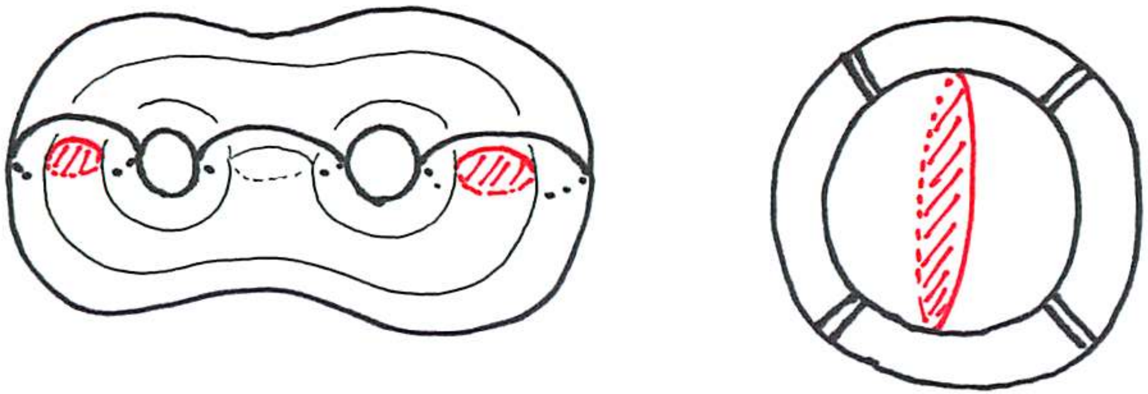
Bridge decomposition of a knot / link K in S^3



$(S^3, K) = (B^3_1, t_1) \underset{S}{\cup} (B^3_2, t_1)$, where (B^3, t_i) : trivial tangle

The punctured sphere $S := \partial B^3_i - t_i$ is also called a bridge sphere

The 3-manifold M $(= S^3 - K)$ is obtained from $S \times [-1, 1]$ by adding 2-handles (and 3-handles)



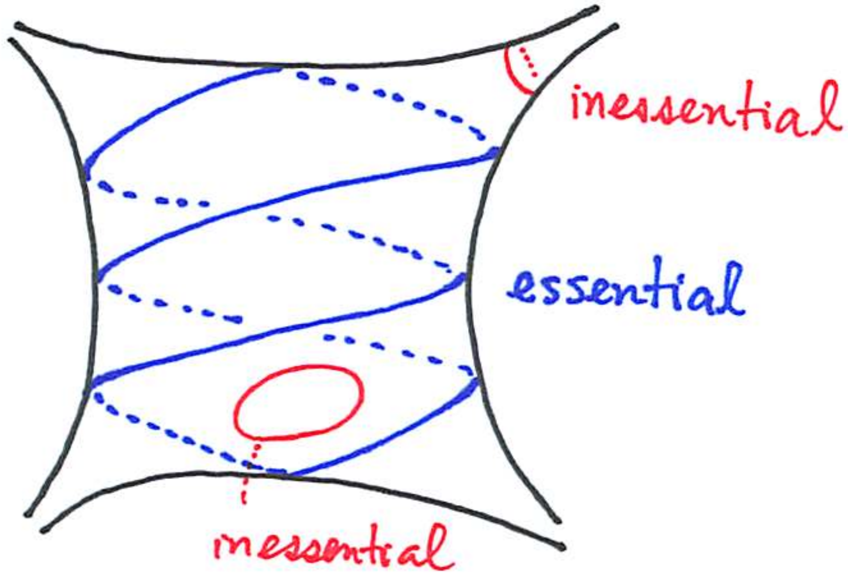
 = core of a 2-handle

In particular $\pi_1(M) \cong \pi_1(S) / \langle\langle \text{2-handles} \rangle\rangle$

$$\mathrm{Hom}(\pi_1(M), PSL(2,\mathbb{C})) \subset \mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{C}))$$

A simple loop α in S is essential

$\Leftrightarrow$ α does not bound a disk nor a once-punctured disk in S



$\mathcal{S} := \{\text{essential simple loops in } S\} / \text{isotopy}$

S : Heegaard surface of a closed orientable 3-manifold M

or

Bridge sphere of a link $K \subset S^3$ with $M := S^3 - K$

Question

For $\alpha \in \mathcal{S}$, an essential simple loop in S;

(1) When is α null-homotopic in M ?

(2) When is α peripheral in M ?

i.e. homotopic to a loop in the peripheral torus $\partial N(K)$.

(3) For $\alpha, \beta \in \mathcal{S}$, when are they homotopic in M ?

Minsky's observation and question

$M = V_1 \cup_S V_2$ Heegaard decomposition

$\mathcal{M}(S) := \{ f : S \xrightarrow[\cong]{} S \text{ homeo} \} / \text{isotopy}$ (extended) mapping class group

$\cup$

$\mathcal{M}(V_i) := \{ f \in \mathcal{M}(S) \mid f \text{ extends to a homeo of } V_i \}$

$\cup$

$\mathcal{M}_0(V_i) := \{ f \in \mathcal{M}(V_i) \mid f_* = id \in \mathrm{Out}(\pi_1(V_i)) \}$

ie $1 \to \mathcal{M}_0(V_i) \longrightarrow \mathcal{M}(V_i) \longrightarrow \mathrm{Out}(\pi_1(V_i)) \to 1$

Consider

$$\mathcal{M}_0(S:M) := \langle \mathcal{M}_0(V_1), \mathcal{M}_0(V_2) \rangle < \mathcal{M}(S)$$

Consider the natural action of $\mathcal{M}(S)$

on the set $\mathcal{S} = \{\text{essential simple loops on } S\}/\sim$

Observation 1

The action of $\mathcal{M}_0(S:M) \subset \mathcal{M}(S)$ on $\mathcal{S}$

preserves the homotopy class in $M = V_1 \underset{S}{\cup} V_2$.

i.e. for any $\alpha \in \mathcal{S}$ and $\gamma \in \mathcal{M}_0(S:M)$,

$\gamma(\alpha) \sim \alpha$ in M.

(Proof) Pick any $\alpha \in \mathcal{S}$.

If $\gamma \in \mathcal{M}_0(V_i) = \mathrm{Ker}(\mathcal{M}(V_i) \longrightarrow \mathrm{Out}(\pi_1(V_i)))$, then

$\gamma(\alpha) \sim \alpha$ in V_i and so in M.

Since $\mathcal{M}_0(S:M) = \langle \mathcal{M}_0(V_1), \mathcal{M}_0(V_2) \rangle$,

we obtain the desired result.

Set

$$\Delta_i := \{\alpha \in \mathcal{S} \mid \alpha \text{ bounds a disk in } V_i\}$$

$$= \{\alpha \in \mathcal{S} \mid \alpha \text{ is null-homotopic in } V_i\}$$

$$Z := \{\alpha \in \mathcal{S} \mid \alpha \text{ is null-homotopic in } M\}$$

Observation 2

$\mathcal{M}_0(S;M)$-orbit of $\Delta_1 \cup \Delta_2$ is contained in Z.

ie $\quad \mathcal{M}_0(S;M) \cdot (\Delta_1 \cup \Delta_2) \subset Z$.

(Proof)

It is obvious that $\Delta_1 \cup \Delta_2 \subset Z$.

Since $\mathcal{M}_0(S;M)$ preserves the homotopy class in M,

we have $\quad \mathcal{M}_0(S:M) \cdot (\Delta_1 \cup \Delta_2) \subset Z$.

Question [Minsky]

When is Z equal to the orbit $\mathcal{M}_0(S;M)\cdot(\Delta_1 \cup \Delta_2)$?

In this talk, we explain the following results.

[Lee-S] Complete answer to a variation of Minsky's question for 2-bridge spheres for 2-bridge links.

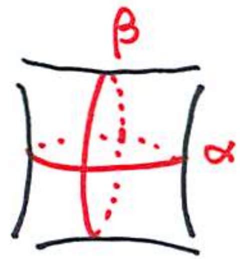
[Lee-S] Application to a variation of McShane's identity for 2-bridge links.

[Ohshika-S] A partial answer in a general setting.

[Akiyoshi-S-Wada-Yamashita] [Akiyoshi-S] Speculations and computer experiments.

- $\mathcal{C}(S)$: curve complex
  - $\mathcal{C}^{(0)}(S) = \mathcal{S}$ = {essential simple loops on S}/isotopy
  - $\alpha, \beta \in \mathcal{S}$ span an edge in $\mathcal{C}(S)$

    $\Leftrightarrow \quad \alpha \cap \beta = \emptyset$

    $|\alpha \cap \beta| = 2$ if $S =$ 

[Ivanov, Korkmaz]

$$1 \to \langle \text{hyper-elliptic involutions} \rangle \to MCG(S) \longrightarrow \mathrm{Aut}(\mathcal{C}(S)) \longrightarrow 1$$

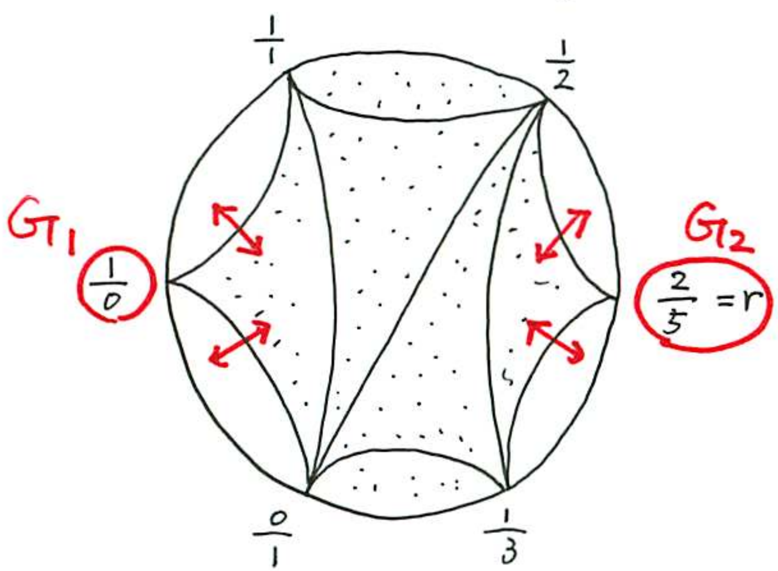
$$\langle \text{hyper-elliptic involutions} \rangle = \begin{cases} \{1\} & \text{generically} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } S = \end{cases}$$

For $M = V_1 \underset{S}{\cup} V_2$, set

$G_i :=$ Image of $\mathcal{M}_0(V_i)$ in $\mathrm{Aut}(\mathcal{C}(S))$

$G := \langle G_1, G_2 \rangle < \mathrm{Aut}(\mathcal{C}(S))$

<u>Example</u> If S is a 2-bridge sphere of a 2-bridge link $K(r)$, then



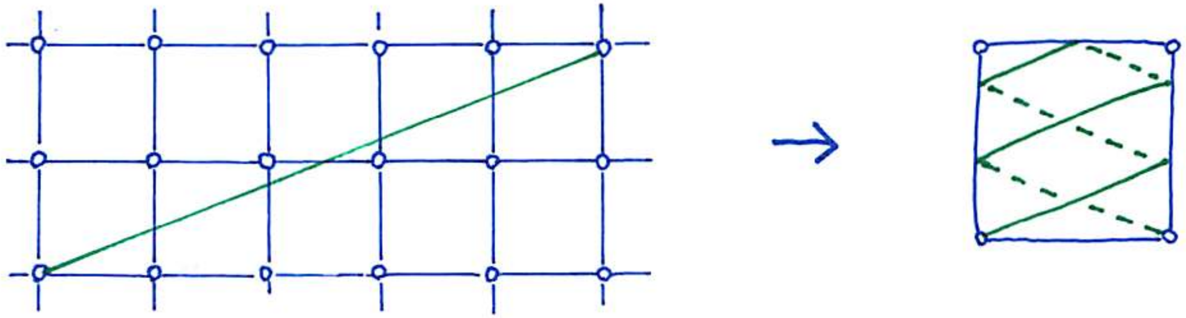
$$(S^3, K(r)) = (B^3, t(\tfrac{1}{0})) \underset{S}{\cup} (B^3, t(r))$$

$$G_1 \cong G_2 \cong D_\infty$$

If $d(\frac{1}{0}, r) \geqq 2$, then $G = G_1 * G_2$.

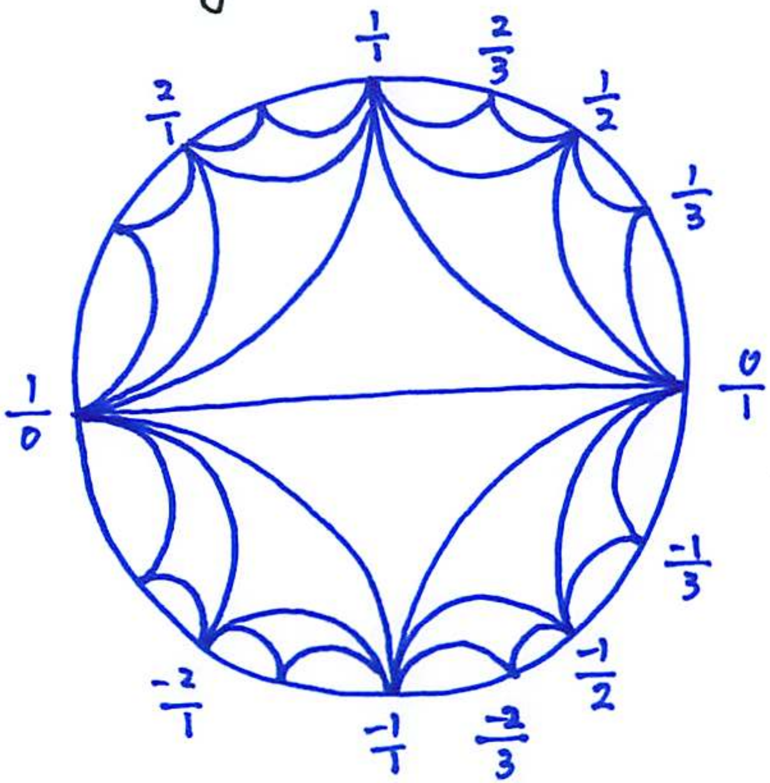
$S := \mathbb{R}^2 - \mathbb{Z}^2 / \langle \pi\text{-rotations around punctures} \rangle$: 4-punctured sphere

(Conway sphere)



$\delta_{2/5}$

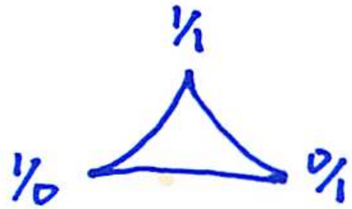
D : Farey tessellation



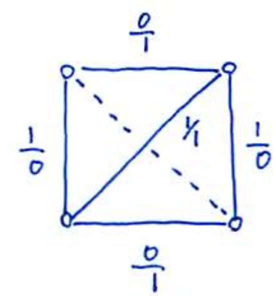
Vertex set of $D = \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\} \ni r$

$\underset{1-1}{\leftrightarrow} \{\text{essential simple loops on } S\} \ni \alpha_r$

$\underset{1-2}{\leftrightarrow} \{\text{essential simple arcs on } S\} \ni \delta_r$

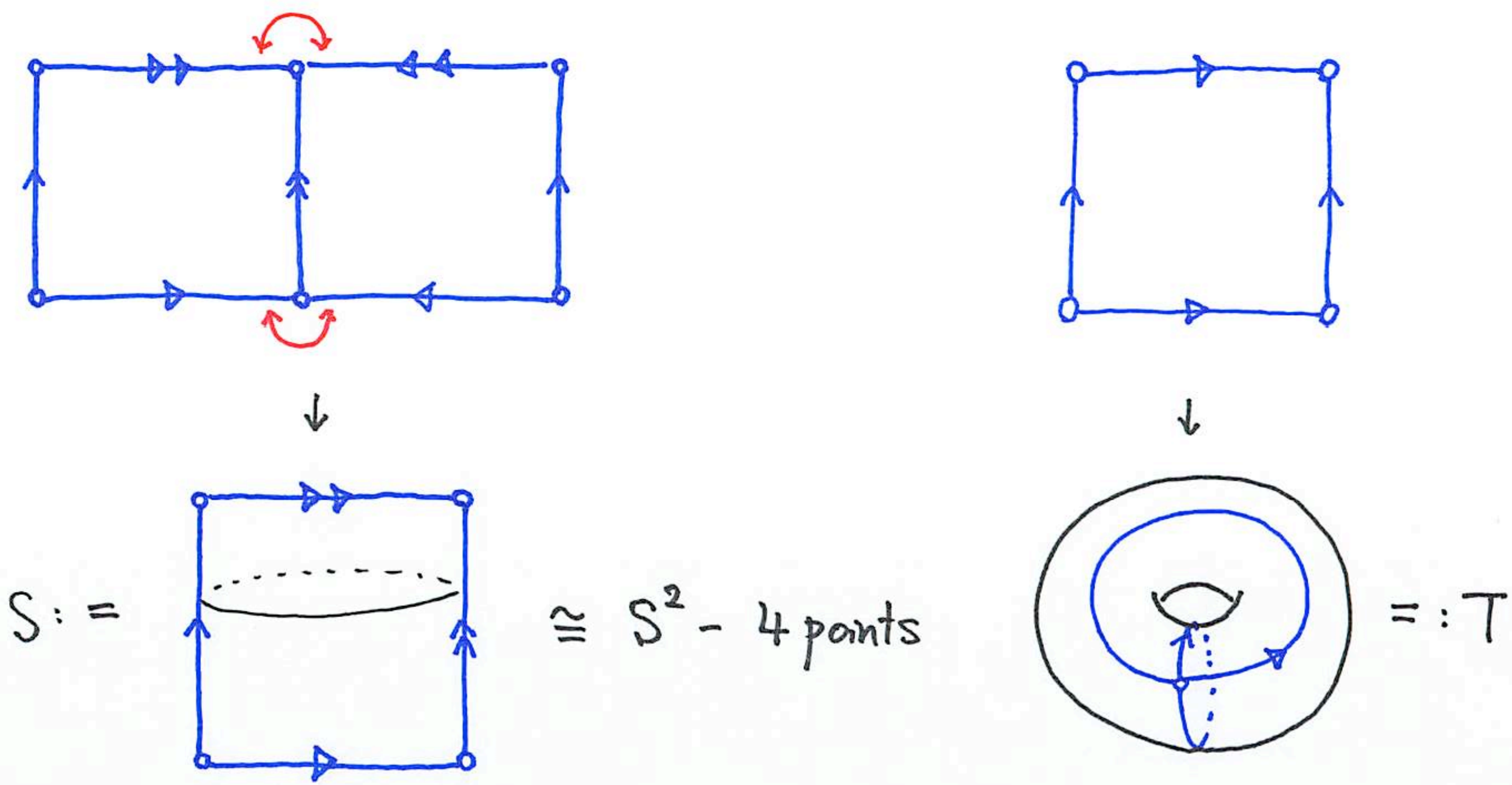


$\leftrightarrow$



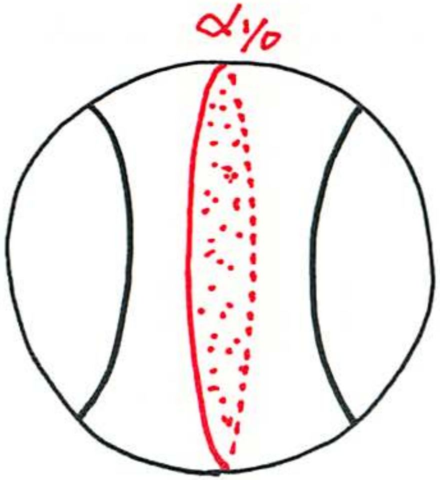
Farey triangle

ideal triangulation of S

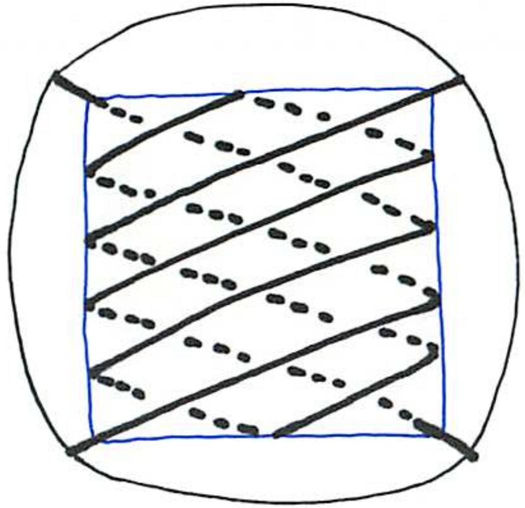


S and T are "commensurable"

Rational tangle $(B^3, t(r))$ of slope r :



$(B^3, t(1/0))$

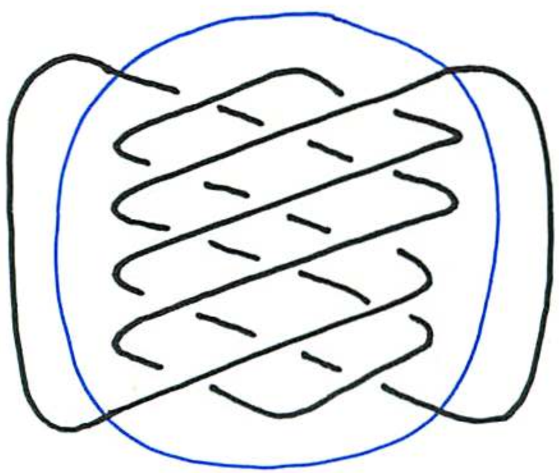


$(B^3, t(2/5))$

$\pi_1(B^3 - t(r))$

$\cong \pi_1(S) / \langle\langle \alpha_r \rangle\rangle$

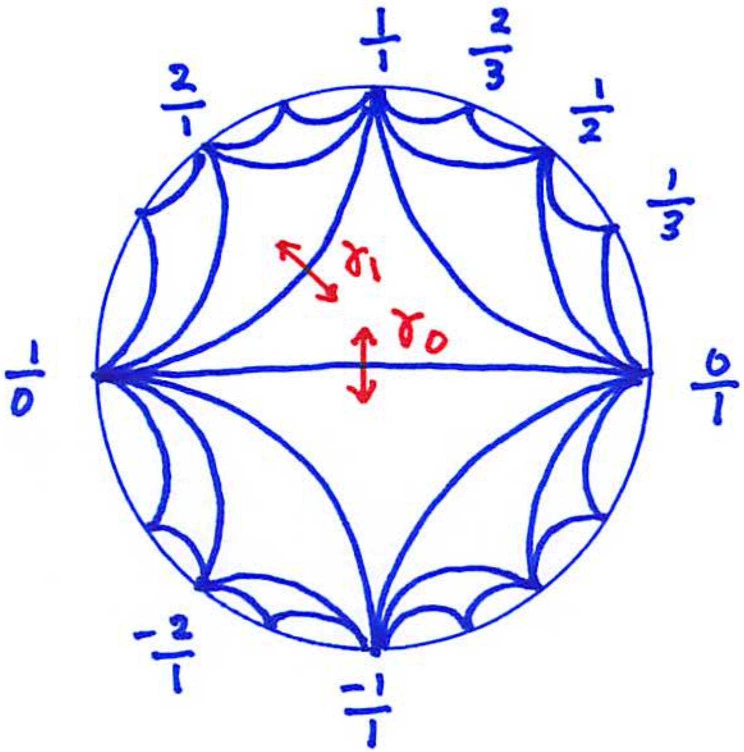
$(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$: 2-bridge link of slope r

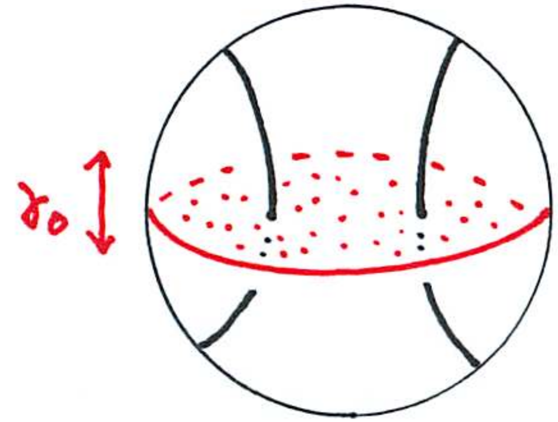


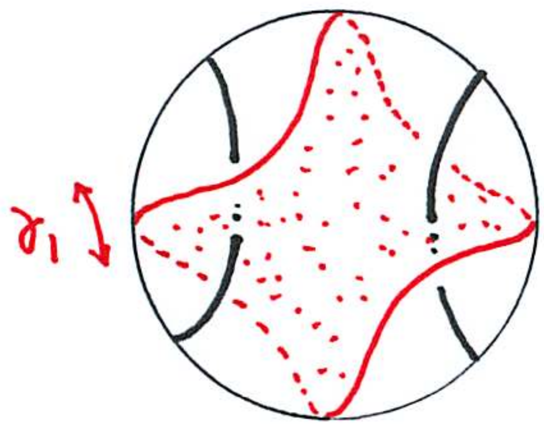
$G(K(r)) := \pi_1(S^3 - K(r))$

$\cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$

$G_1 = \langle \gamma_0, \gamma_1 \rangle \cong D_\infty$







Theorem [Lee-S]

For a 2-bridge sphere S of a 2-bridge link $K(r)$, the following hold.

(1) $\mathcal{Z} = G \cdot (\Delta_1 \cup \Delta_2)$

(2) Suppose $K(r)$ is hyperbolic (ie $d(\infty, r) \geqq 3$) and is not a twist knot (ie $r \not\equiv \frac{\pm n}{2n+1}$ in $\mathbb{Q}/\mathbb{Z}$).
Then any simple loop $\alpha \in \mathcal{S} - \mathcal{Z}$ is non-peripheral.

(3) Suppose $K(r)$ is hyperbolic (ie $d(\infty, r) \geqq 3$) and is not a Whitehead link (ie $r \not\equiv \pm\frac{3}{8}$ in $\mathbb{Q}/\mathbb{Z}$).
Then two simple loops α, β in $\mathcal{S} - \mathcal{Z}$ are homotopic in $S^3 - K(r)$, iff α and β belong to the same G-orbit.

(Idea of proof)

Apply the small cancellation theory to the 2-generator & 1-relator presentation

$$\pi_1(S^3 - K(r)) \cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$$

$$\cong \left(\pi_1(S) / \langle\langle \alpha_\infty \rangle\rangle \right) / \langle\langle \alpha_r \rangle\rangle$$

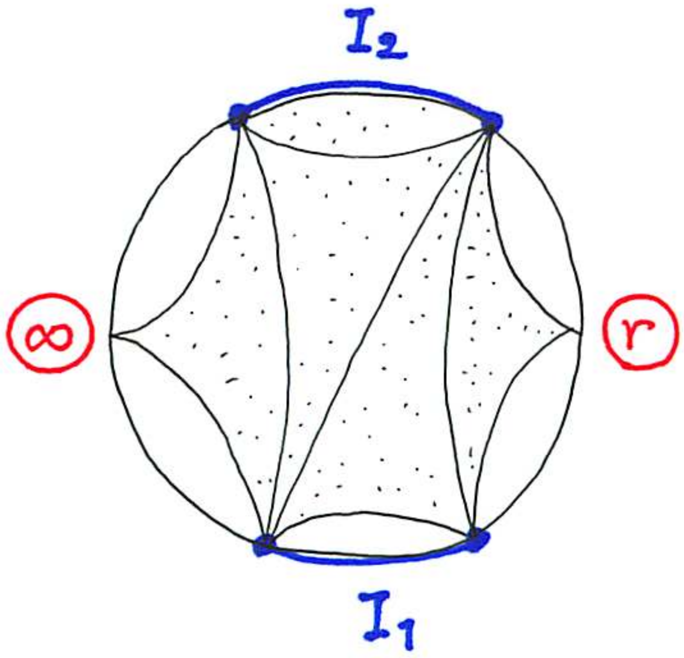
$\cong \pi_1($  $) / \langle\langle \alpha_r \rangle\rangle$

(Intuition behind the proof)

If $s \in I_1 \cup I_2$, then

α_s is "far from"

the relators α_∞ and α_r.



Corollary For a 2-bridge link $K(r)$ with $d(\infty, r) \geqq 4$

(1) $G = \langle G_1, G_2 \rangle$ is the free product $G_1 * G_2$.

(2) The limit set $\Lambda(G)$ of $G \curvearrowright PML(S)$ is a measure 0 Cantor set, and we have $\Lambda(G) = \overline{Z}$, where $Z = \{ \alpha \in \mathcal{S} \mid \alpha \sim 1 \text{ in } S^3 - K(r) \}$

(3) The domain of discontinuity $\Omega(G)$ has the full measure and any simple loop in $\Omega(G)$ is not null-homotopic in $S^3 - K(r)$.

(4) For two simple loops $\alpha, \beta \in \Omega(G)$, $\alpha \sim \beta$ in $S^3 - K(r)$ iff α and β belong to the same G-orbit.

Question Do these results hold in a general setting?

Theorem [Bowditch - Ohshika - S]

There is a constant K_0, depending only on S, st

for any Heegaard or bridge decomposition $M = V_1 \underset{S}{\cup} V_2$

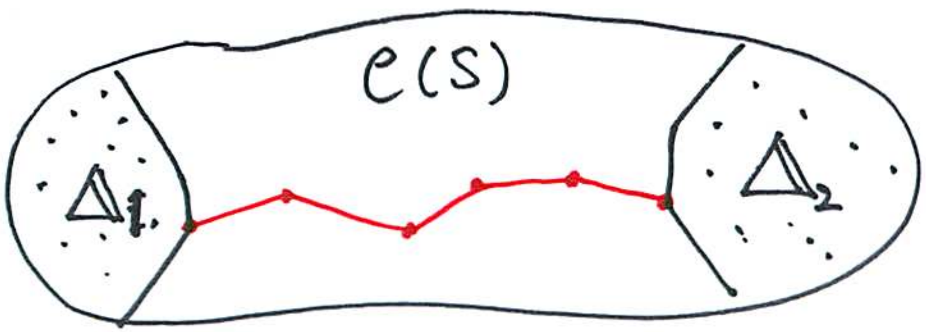
with Hempel distance $d(V_1, V_2) \geqq K_0$,

we have $G_\Gamma = G_{\Gamma_1} * G_{\Gamma_2}$.

Hempel distance

$$d(V_1, V_2) := d_{e(S)}(\Delta_1, \Delta_2), \text{ where}$$

$$\Delta_i := \{\text{meridians of } V_i\} \quad (i = 1, 2)$$



Theorem [Ohshika - S]

For any closed orientable surface S of genus $\geqq 2$ and any $\varepsilon > 0$, there is a constant $K_0 = K_0(S, \varepsilon)$, st:

If M is a hyperbolic manifold with

$$\mathrm{inj}(M) := \frac{1}{2} \min \{ \ell(\gamma) \mid \gamma \text{ is a closed geodesic in } M \} \geqq \varepsilon$$

and if $M = V_1 \underset{S}{\cup} V_2$ is a Heegaard decomposition with Hempel distance $\geqq K_0$,

then there is a nonempty open set $\mathcal{O}$ in $\mathcal{PML}(S)$, st

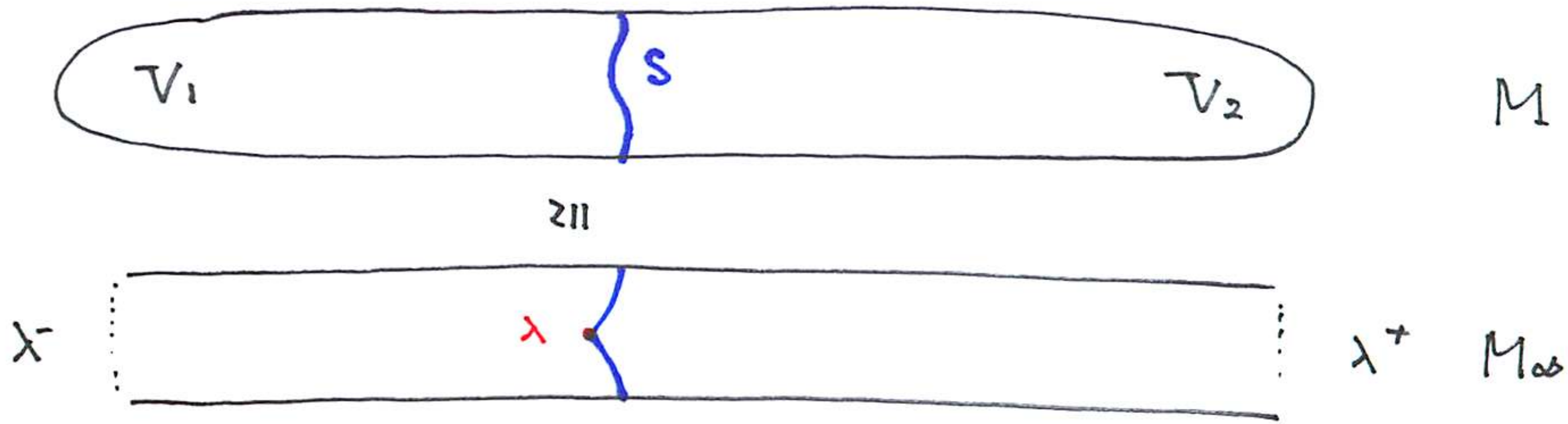
(i) $\forall \alpha \in \mathcal{O} \cap \mathcal{S}$, $\alpha \not\sim 1$ in M

(ii) $\forall \alpha, \beta \in \mathcal{O} \cap \mathcal{S}$ distinct elements, $\alpha \not\sim \beta$ in M

Cor The action $\hat{\Gamma} \curvearrowright \mathcal{PML}(S)$ has a non-empty domain of discontinuity.

Idea (Brock - Minsky - Namazi - Souto)

If the Hempel distance is large, then the hyperbolic manifold $M = V_1 \cup_S V_2$ looks like a hyperbolic manifold M_∞ homeomorphic to $S \times \mathbb{R}$ whose ends are geometrically infinite.



Let $f : S \to M_\infty$ be a pleated surface lying in the "central part" with a depth 0 pleating lamination λ. Then a small nbd O of $[\lambda] \in \mathcal{PML}(S)$ satisfied the desired conditions.

Speculations and Questions

Suppose $M = V_1 \cup_S V_2$ admits a complete hyperbolic structure.

Then

$$
\begin{tikzcd}
\pi_1(M) \arrow[r, hook] & \mathrm{Isom}^+ \mathbb{H}^3 = PSL(2,\mathbb{C}) \\
\pi_1(S) \arrow[u, two heads] \arrow[ur, "\rho_M"'] &
\end{tikzcd}
$$

ρ_M non-faithful, discrete image

$$R_S = \mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{C})) / \mathrm{conj}$$

$$\cup$$

$$\mathcal{D}_S = \{ \rho \in R_S \mid \rho \text{ has discrete image} \}$$

$$\cup$$

$$\mathcal{DF}_S = \{ \rho \in R_S \mid \rho : \text{discrete faithful} \}$$

$$\shortparallel$$

$\overline{\mathcal{QF}_S}$, where $\mathcal{QF}_S = \{$ quasifuchsian representations $\}$

ρ_M is an isolated point of R_S by Mostow rigidity theorem.

Question

(1) Is there a natural path in R_S joining ρ_M with $\mathcal{DF}_S$?

(2) Can we find and prove properties of ρ_M by using the natural path ?

(3) How are ρ_M's distributed in R_S ?

Partial Answer for 2-parabolic generator groups

$$G_\omega := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle \subset PSL(2, \mathbb{C})$$

$$\mathcal{DF} = \{ \omega \in \mathbb{C} \mid G_\omega \text{ is a free Kleinian group} \} = \overline{\text{Riley slice}}$$

[Akiyoshi - S - Wada - Yamashita]

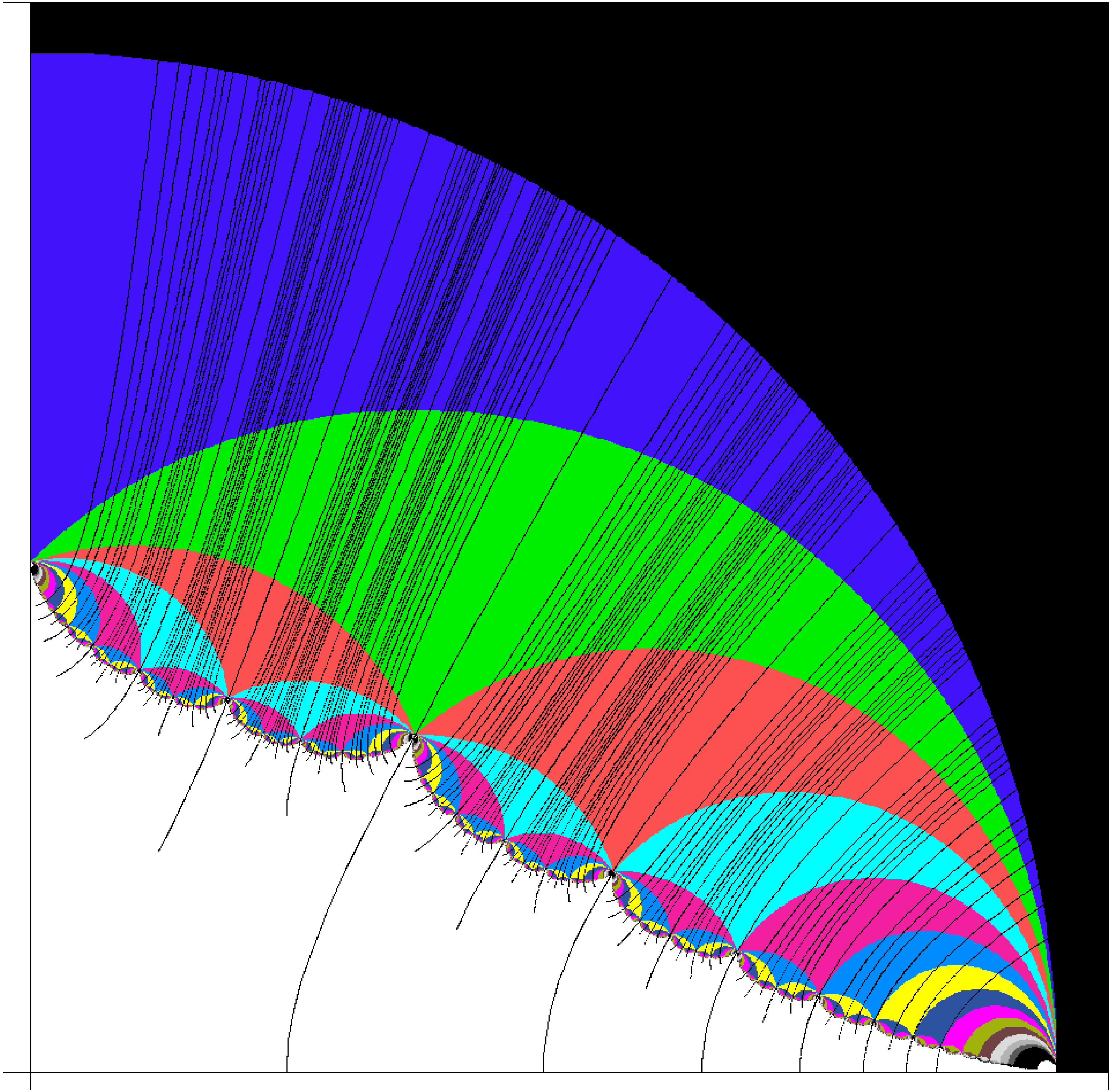
Each 2-bridge link has a natural path to $\mathcal{DF}$.

[Akiyoshi - S]

Representations in the paths seem to have nice properties.

[Ohshika - Miyachi] $\partial(\mathcal{QF})$ is a Jordan circle.

[Martin - S] $\partial(\mathcal{QF})$ = Accumulation set of $\{$2-bridge links$\}$



ありがとうございました

Thank you very much!

Happy Birthday to Sasha !!