On the structure of volume set of hyperbolic polyhedra

Konstantin Shmelkov

Novosibirsk State University

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Statement of problem

Consider the class of compact right-angled hyperbolic polyhedra and find exact volumes of those having a rather small volume.

Definition

A polyhedron is called *right-angled* if all its dihedral angles are right. In this case its face angles are also right.

Corollary (Mostow rigidity)

A volume of a hyperbolic polyhedron is its topological invariant.

Corollary (Thurston-Jorgensen)

The set of volumes of the hyperbolic polyhedra is well-ordered.

Combinatorial approach

Corollary (Andreev's theorem)

A hyperbolic polyhedron with non-obtuse dihedral angles can be is defined by its 1-skeleton and dihedral angles up to isometry.

Theorem (Steinitz's)

A graph can be 1-skeleton of a convex polyhedron in $\mathbb{E}^3$ *if and only if the graph is 3-connected planar simple graph. We will call it polyhedral graph.*

Andreev's theorem for compact polyhedra

Theorem (Andreev, 1970)

A polyhedral graph can be 1-skeleton of a compact right-angled hyperbolic polyhedron if and only if the graph is 3-valent and cyclically 5-connected. (denote this class by $\mathcal{P}$).

Definition

A graph is called *cyclically k-connected* if at least k edges have to be removed to split it into two connected components that **both** have a cycle.

Results of T. Inoue (2008)

Introduce two operations on 3-valent cyclically 5-connected graphs (class $\mathcal{P}$):

$$*\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P} \text{ — composition}$$

$$'\colon \mathcal{P} \to \mathcal{P} \text{ — edge surgery}$$

Theorem (Inoue, 2008)

Let $P_1, P_2 \in \mathcal{P}$ and $P_3 = P_1 * P_2$. Then $P_3 \in \mathcal{P}$ and volumes of corresponding right-angled hyperbolic polyhedra $\widehat{P_1}$, $\widehat{P_2}$, $\widehat{P_3}$ are related by the following inequalities :

$$\mathrm{vol}(\widehat{P_3}) \geqslant \mathrm{vol}(\widehat{P_1}) + \mathrm{vol}(\widehat{P_2})$$

Let $P \in \mathcal{P}$. Then $P' \in \mathcal{P}$ and volumes of corresponding right-angled hyperbolic polyhedra $\widehat{P}$ and $\widehat{P'}$ are related by the following inequality :

$$\mathrm{vol}(\widehat{P'}) > \mathrm{vol}(\widehat{P})$$

Löbell polyhedra

Definition

Löbell polyhedron $R(n)$ ($n \geqslant 5$) is the right-angled hyperbolic polyhedron with $(2n+2)$ faces, two of which (viewed as the upper and lower bases) are regular n-gons, while the lateral surface is given by $2n$ pentagons.

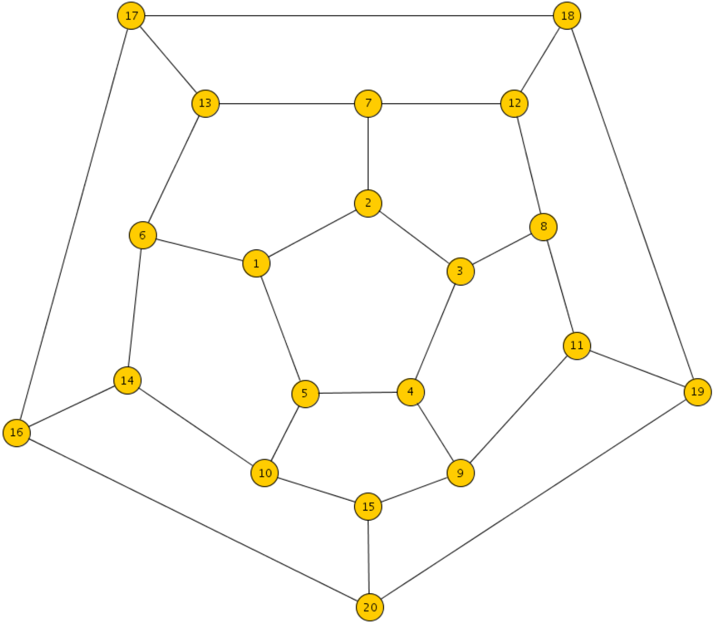
Theorem (Vesnin, 1998)

$$\mathrm{vol}(R(n)) = \frac{n}{2}\left(2\Lambda\left(\theta_n\right) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \Lambda\left(\theta_n - \frac{\pi}{n}\right) - \Lambda\left(2\theta_n - \frac{\pi}{2}\right)\right)$$

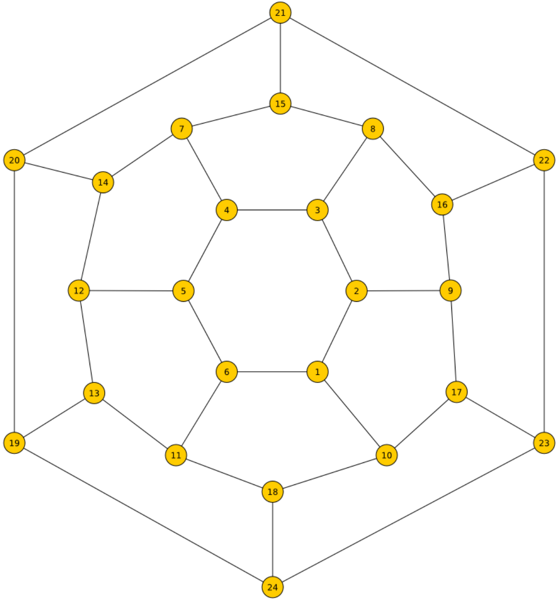
$$\theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\frac{\pi}{n})}\right)$$

$$\Lambda(z) = -\int_0^z \log|2\sin(t)|\,\mathrm{d}t$$

Löbell polyhedron $R(5)$



Löbell polyhedron $R(6)$



Classification theoram

Theorem (Inoue, 2008)

Each right-angled compact hyperbolic polyhedron can be obtained from one of the Löbell polyhedra by successive application of a finite number of the composition $*$ *or the edge surgery* $'$*.*

Corollary

The smallest-volume polyhedron in $\mathcal{P}$ *is the dodecahedron (*$R(5)$*). The next one by volume is* $R(6)$*.*

Statement of problem (ideal case)

Consider the class of *ideal* right-angled hyperbolic polyhedra and find exact volumes of those having a rather small volume.

Theorem (Rivin, 1992)

A polyhedral graph can be 1-skeleton of an ideal right-angled hyperbolic polyhedra if and only if the graph is 4-valent and cyclically 6-connected (denote this class by $\mathcal{A}$).
Moreover, the geometrical realization is unique up to isometry of $\mathbb{H}^3$.

Classification theorem for ideal polyhedra

Theorem (McKay, Brinkmann and others, 2005)

Each 4-valent cyclically 6-connected polyhedral graph can be obtained from one of antiprisms' 1-skeletons by successive application of a finite number of edge twisting.

Definition

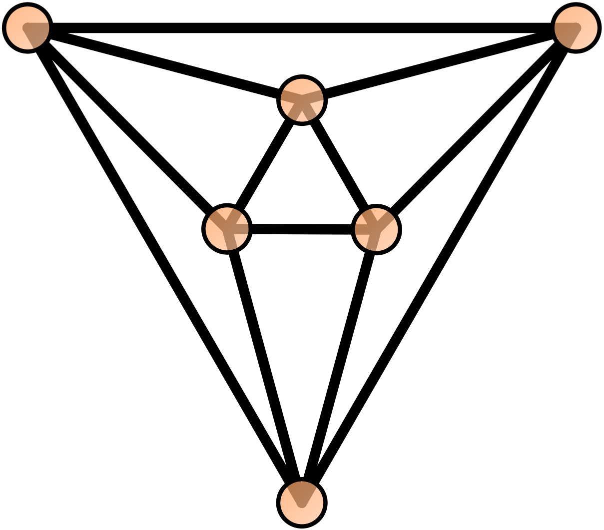
Antiprism $A(n)$ is the ideal right-angled hyperbolic polyhedron with $(2n+2)$ faces, two of which (viewed as the upper and lower bases) are regular n-gons, while the lateral surface is given by $2n$ triangles.

Theorem (Thurston, 1980)

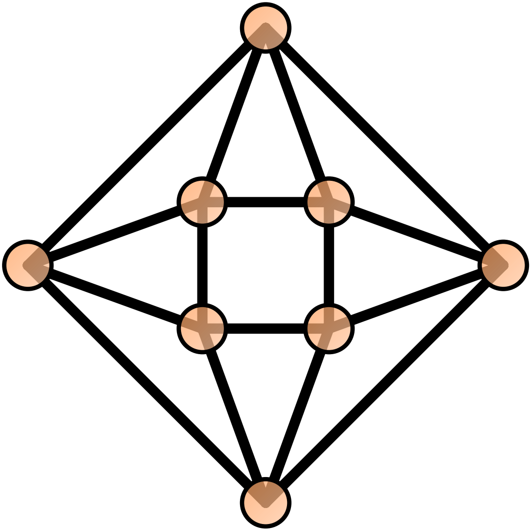
$$\mathrm{vol}(A(n)) = 2n\left(\Lambda\left(\frac{\pi}{4}+\frac{\pi}{2n}\right)+\Lambda\left(\frac{\pi}{4}-\frac{\pi}{2n}\right)\right)$$

$$\Lambda(z) = -\int_0^z \log|2\sin(t)|\,\mathrm{d}t$$

Samples of antiprisms: $A(3)$

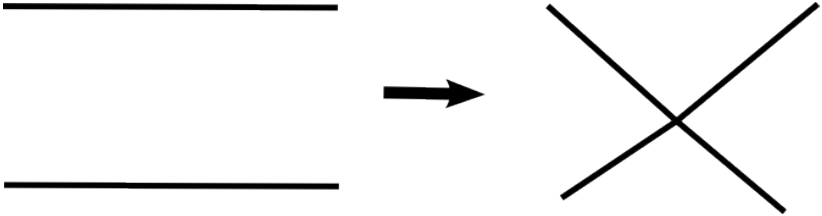


Samples of antiprisms: $A(4)$

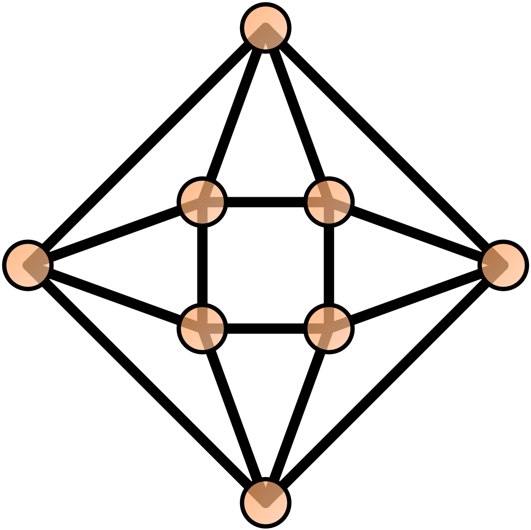


Edge twisting : schema

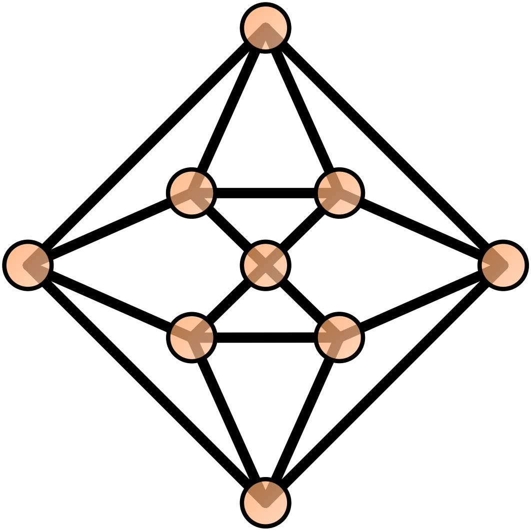
$^+ : \mathcal{A} \to \mathcal{A}$ — edge twisting



Edge twisting on $A(4)$: Before



Edge twisting on $A(4)$: After



Volume relations

Theorem (Shmelkov, 2011)

Let P_1 is the result of an edge twisting appliction on a 4-valent cyclically 6-connected polyhedron P_0 : $P_1 = P_0^+$. Then volumes of corresponding ideal right-angled hyperbolic polyhedra $\widehat{P_0}$ and $\widehat{P_1}$ are related by the following inequality :

$$\mathrm{vol}(\widehat{P_0}) < \mathrm{vol}(\widehat{P_1})$$

Corollary

The smallest-volume polyhedron in $\mathcal{A}$ is the octahedron ($A(3)$). The next one by volume is $A(4)$.

List of smallest-volume ideal polyhedra

Theorem (Shmelkov, 2011)

Any ideal right-angled hyperbolic polyhedron whose volume is at most $\mathrm{vol}(A(9)) = 15.9333855\ldots$ *is isometric to a polyhedron from the following list and has an enlisted volume :*

- $\mathrm{vol}(A(3)) \approx 3.6638625$
- $\mathrm{vol}(A(4)) \approx 6.0230460$
- $\mathrm{vol}(A(4)^{+}) \approx 7.3277250$
- $\mathrm{vol}(A(5)) \approx 8.1378850$
- $\mathrm{vol}(A(4)^{++}) \approx 8.6124150$
- $\mathrm{vol}(A(5)^{+}) \approx 9.6869085$
- $\mathrm{vol}(A(4)^{+++}) \approx 10.149416$
- $\mathrm{vol}(A(6)) \approx 10.1494160$
- $\mathrm{vol}(A(5)_1^{++}) \approx 10.8060025$
- $\mathrm{vol}(A(5)_4^{++}) \approx 10.9915870$
- $\mathrm{vol}(A(5)_6^{++}) \approx 10.9915870$
- $\mathrm{vol}(A(5)_5^{++}) \approx 11.1362960$
- $\mathrm{vol}(A(5)_2^{++}) \approx 11.1362960$
- $\mathrm{vol}(A(5)_3^{++}) \approx 11.4472075$
- $\mathrm{vol}(A(4)_1^{++++}) \approx 11.8017475$
- ...
- $\mathrm{vol}(A(9)) \approx 15.9333855$