A Dilogarithm Identity on Moduli Spaces of Curves

Ser Peow Tan 陈史标
National University of Singapore

Joint w/ Feng Luo, 罗锋 Rutgers Univ.

PLAN

0 : Introduction

1 : Identities, generalizations & applications

2: Sketch of proofs of Identities of Basmajian, McShane, & Bridgeman

3: Sketch of proof for identity for closed surfaces (Luo-T).

Introduction....

- Give a new identity for (closed) hyperbolic surfaces.

- Relate to various identities for surfaces with boundary by Basmajian, McShane, Bridgeman & various others.

- Discuss some generalizations & applications

- Unified point of view & sketch proofs for the various identities

- Sketch proof of main result.

1/ Identities

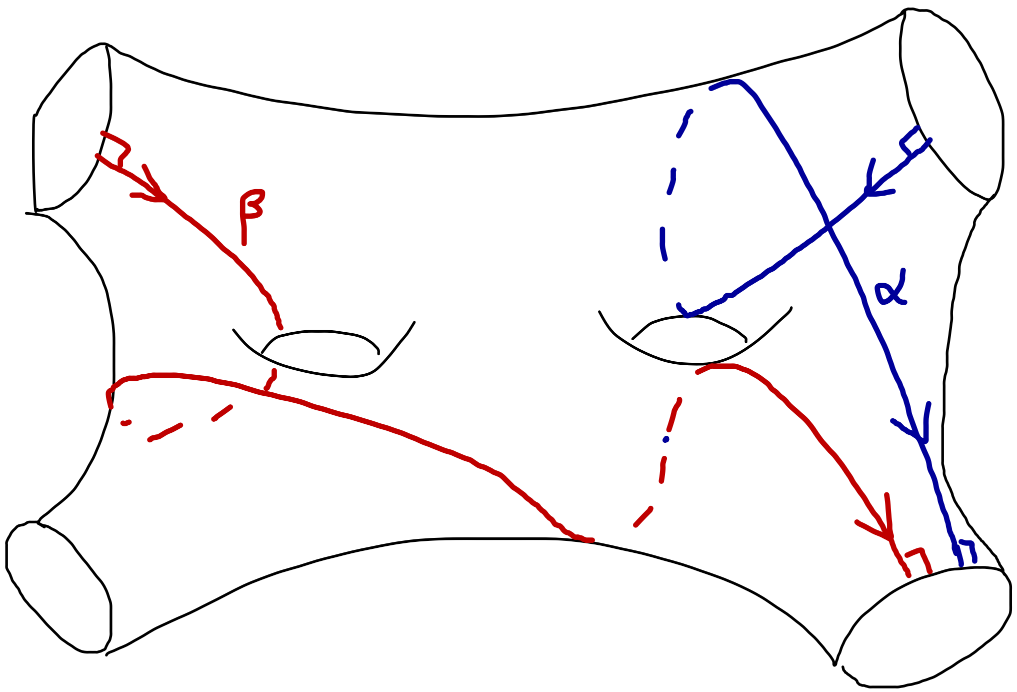
A. Thm (Basmajian '93 AJM)

$$\sum_{\alpha} B(|\alpha|) = L$$

over all oriented orthogeodesics

Total length of ∂S.

$$B(x) = 2\log\coth\frac{x}{2} = 2\sinh^{-1}\left(\frac{1}{\sinh x}\right)$$

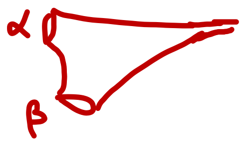


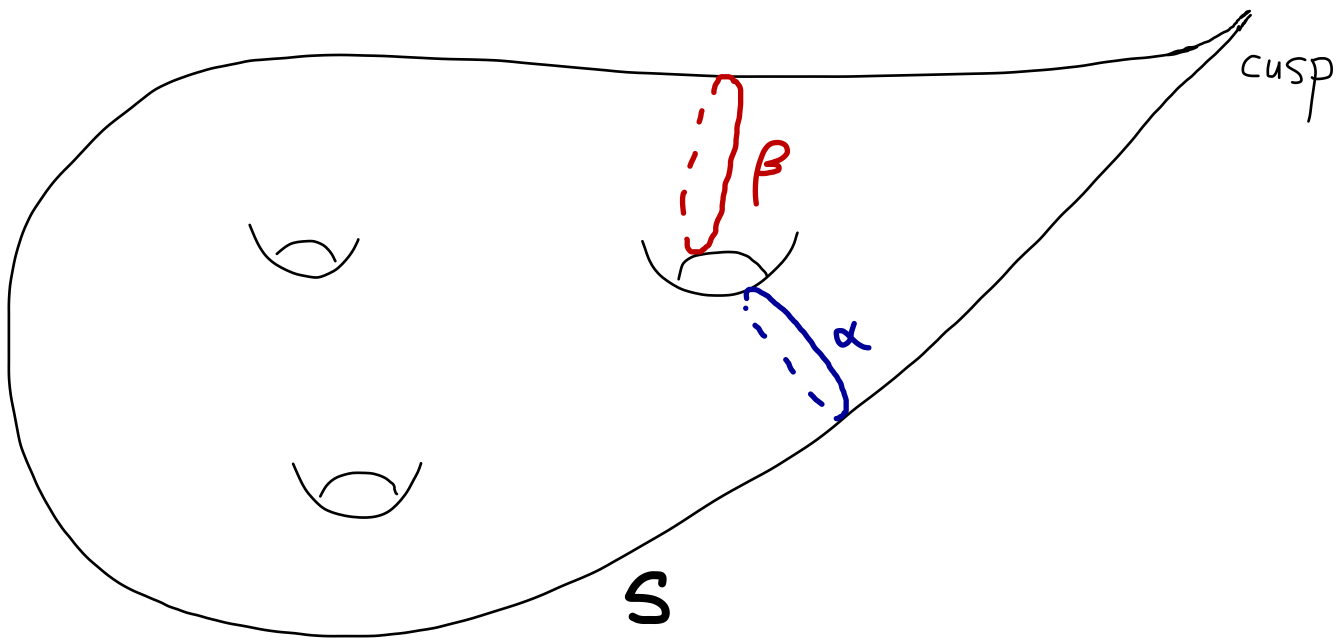
B1 Thm (McShane '91, 98)

$$2\sum_{\alpha,\beta} \frac{1}{1+\exp\left(\frac{|\alpha|+|\beta|}{2}\right)} = 1.$$

$$M(x,y) = \frac{2}{1+\exp\left(\frac{x+y}{2}\right)}$$

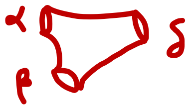
over all





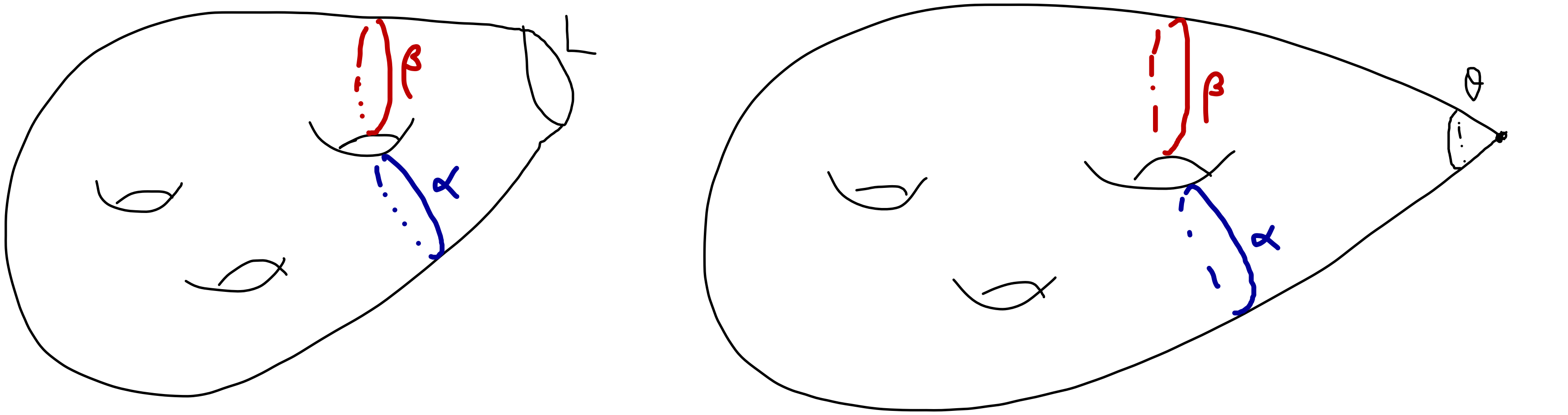
B2 Thm (Mirzakhani '08, T-Wong-Zhang '07)

$$2\sum_{\alpha,\beta} G\left(\frac{L}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) = L \qquad G(x,y,z) = \log\left(\frac{e^{x}+e^{y+z}}{e^{-x}+e^{y+z}}\right)$$

over all 

length of $\delta = \partial S$

Here $\partial S = \delta$ with $|\delta| = L$ & δ may be geodesic boundary or cone pt. of $\measuredangle$ $0 \leq \theta \leq \pi$.



C Thm (Bridgeman '11)

$$\sum_{\alpha} Br(|\alpha|) = 4\pi^2 |\chi(S)|$$

↑ over all oriented orthogeodesics

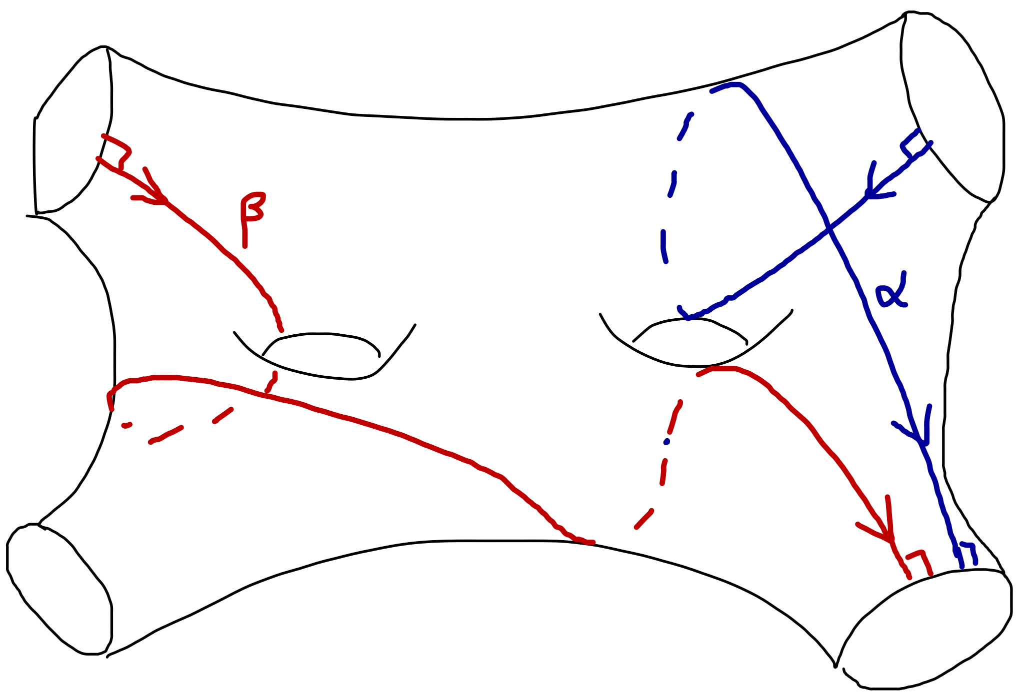
Vol $(T_1(S))$

$Br(x) = 4\,\mathcal{L}\left(\frac{1}{\cosh^2 \frac{x}{2}}\right)$

where $\mathcal{L}(x)$ is Roger's dilog function,

$\mathcal{L}(0)=0, \quad -2\mathcal{L}'(x) = \frac{\log x}{1-x} + \frac{\log(1-x)}{x},$

$0 \le x \le 1$



QUESTION :

IS THERE AN IDENTITY FOR CLOSED HYPERBOLIC SURFACES ?

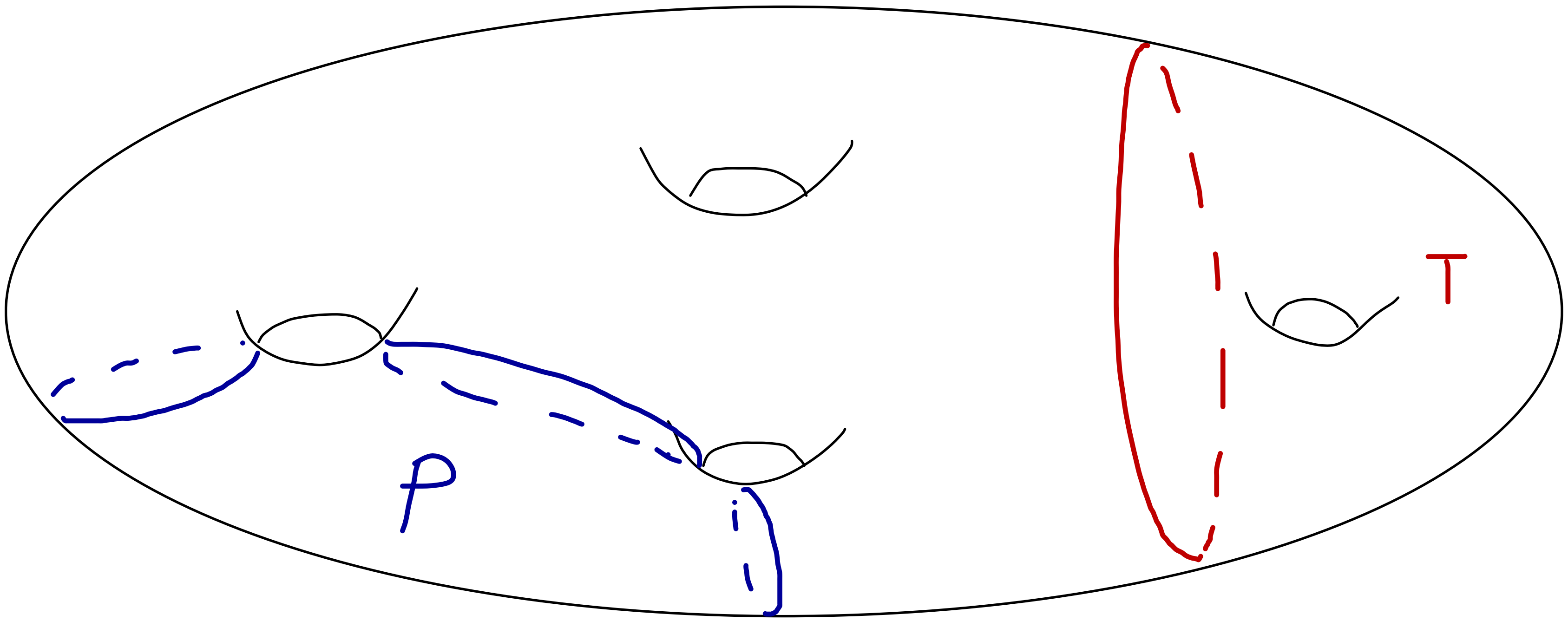
D Thm (Luo-T, '11)

$$\sum_{P} f(P) + \sum_{T} g(T) = -4\pi^2 \chi(S)$$

over all p.o.p.

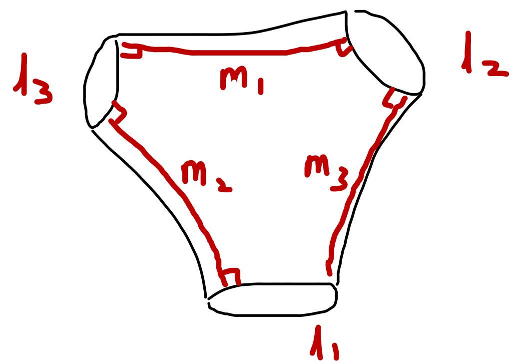
over all one-hded torus

Vol $(T_1(S))$



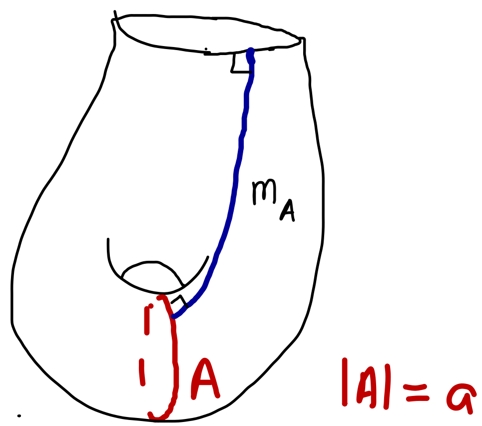
$$f(P) = 4\sum_{i\neq j}\left[2\mathcal{L}\left(\frac{1-x_i}{1-x_i y_j}\right) - 2\mathcal{L}\left(\frac{1-y_j}{1-x_i y_j}\right) - \mathcal{L}(y_j) - \mathcal{L}\left(\frac{(1-x_i)^2 y_j}{(1-y_j)^2 x_i}\right)\right]$$

$$x_i = e^{-l_i}, \quad y_j = \tanh^2\left(\frac{m_j}{2}\right)$$



$$g(T) = 4\pi^2 + 8\sum_{A}\left[2\mathcal{L}\left(\frac{1-x_A}{1-x_A y_A}\right) - 2\mathcal{L}\left(\frac{1-y_A}{1-x_A y_A}\right) - 2\mathcal{L}(y_A) - \mathcal{L}\left(\frac{(1-x_A)^2 y_A}{(1-y_A)^2 x_A}\right)\right]$$

$$x_A = e^{-a}, \quad y_A = \tanh^2\left(\frac{m_A}{2}\right)$$



Generalizations & Applications

A - Basmajian - higher dimensions

B - McShane - geodesic boundaries, cone singularities (Mirzakhani, T-Wong-Zhang)

punctured surface bundles over circle (Bowditch, Akiyoshi-Miyachi-Sakuma),

quasi-fuchsian rep^ns (Bowditch), hyperbolic 3-mfds from Dehn Surgery (T-Wong-Zhang),

closed genus 2 surface (McShane), Hitchin components (Labourie-McShane), complex

hyperbolic rep^ns (Kim-Kim-T), Do-Norbury, 2-bridge knots (Lee-Sakuma) ....

Applications - (Mirzakhani) WP-vol. of moduli space, asymptotics of growth of lengths of simple closed geodesics, Kontsevich-Witten Thm etc...

$\subseteq$ Bridgeman - higher dimensions, (Bridgeman-Kahn), applications to lower bounds for volumes.

D - Luo - T: An number (including none) of boundary components.

Non-orientable surfaces.

Possible applications to the counting of embedded pairs of pants.

2/ Sketch of proofs.

Idea: (X, μ), with $\mu(X) < \infty$.

A. Find interesting geometric & measure theoretic decompositions of X.

$$X = Z \sqcup \bigsqcup_i W_i$$

$\mu(Z) = 0$

$$\Rightarrow \mu(X) = \sum_i \mu(W_i).$$

B. Compute $\mu(W_i)$ to obtain an identity.

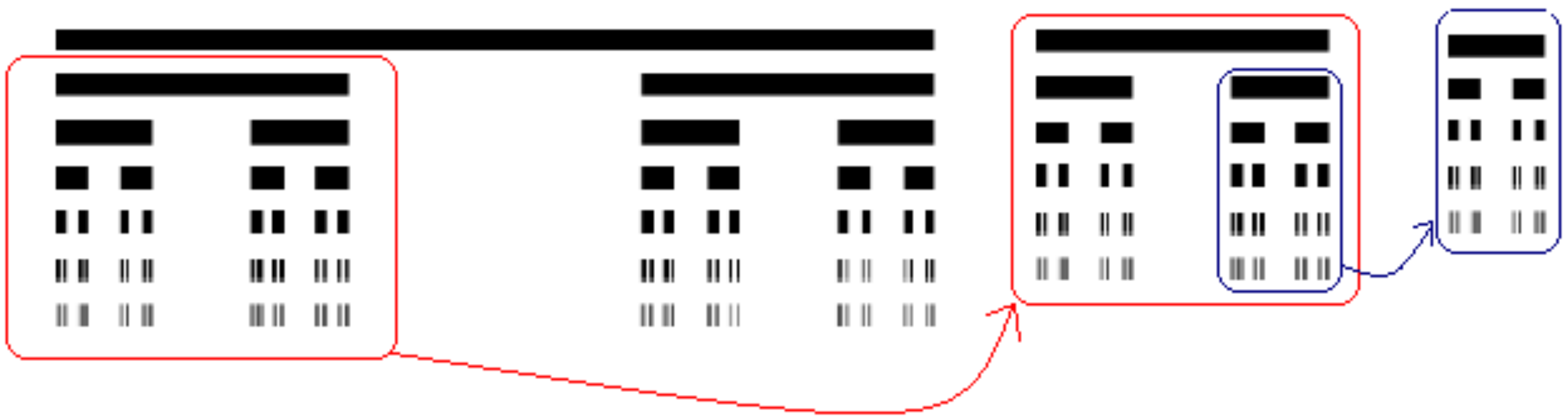
Example 0:

$X = [0,1]$, $Z =$ Cantor Set obtained from removing middle $\frac{1}{3}$ construction.

$W_i =$ complementary intervals – ordered by length & from left to right.

$$\mu(X) = \underbrace{\mu(Z)}_{=0} + \sum_i \mu(W_i)$$

$$\Rightarrow 1 = 0 + \frac{1}{3} + \frac{2}{3^2} + \ldots + \frac{2^i}{3^{i+1}} + \ldots = \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i}.$$



In fact, Basmajian & McShane are interesting variations of example 0!

Example 1. (Basmajian).

$X = \{$ unit tangent vectors on ∂S $\perp$r to $\partial S\} \cong \partial S$.

$\mu(X) = L = \text{length}(\partial S)$.

$v \in X$, $\gamma(v)$ = maximal geodesic in S
obtained by exponentiating v.

$\gamma(v) : [0, T] \to S$, $\gamma(v)(t) = \exp(tv)$. (laser ray starting from v)

Q: When is $T = \infty$ & how to decompose X ?

Answer: $Z = \{ v \in X \mid T = \infty \}$ has measure zero (limit set has measure 0)

Every $v \in X \setminus Z$ is homotopic rel. to ∂S to a unique orthogeodesic α_i. $\{\alpha_i\}$ set of orthogeodesics on S.

Define $W(\alpha_i) = \{ v \in X \mid \gamma(v) \sim \alpha_i \}$, then $X = Z \sqcup \bigsqcup_i W(\alpha_i)$

$\Rightarrow L = \sum_i \mu(W(\alpha_i))$

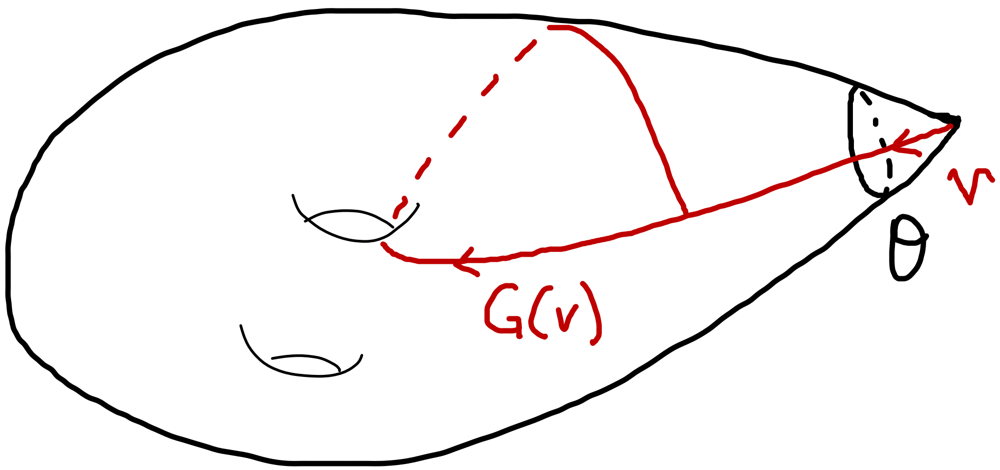
& $\mu(W(\alpha_i)) = 2 \log \coth \frac{|\alpha_i|}{2}$ by elementary hyperbolic geometry calculations.

Example 2 (McShane)

(For simplicity), suppose ∂S has 1 cone pt. with $\measuredangle\ \theta$, $0<\theta<\pi$.

Again, $X = \{$unit tangent vectors at cone pt.$\}$, $\mu(X) = \theta$.

! Instead of lasers, build walls by exponentiating v.



$G(v) : [0,T] \to S$ s.t. $G(v)(t) = \exp(tv)$ & $G(v)$ is an embedding on $[0,T)$.

$Z = \{ v \in X \mid G(v) \text{ is infinite, i.e. } T = \infty \}$.

Again $\mu(Z) = 0$ (by variation of Birman-Series)

For $v \in X \setminus Z$, "stability", nearby v generate 'homotopic' $G(v)$.

Namely: if $v \in X \setminus Z$, $N(\partial S \cup G(v))$ is a pair of pants P which can be made "geometric", so $\partial P = \partial S \cup \alpha \cup \beta$ where α, β disjoint simple closed geodesics.

Let $\{P\}$ be set of embedded p.o.p.'s in S whose boundary includes ∂S, & let $W(P) = \{ v \in X \mid N(G(v) \cup \partial S) \sim P \}$, then $X = Z \sqcup \bigsqcup_P W(P)$.

Computing $\mu(W(P))$ gives $G(x,y,z)$. $\square$ Note: computation is 'localized' at P.

Example 3 (Bridgeman)

Interior tangent vectors

! $X = T_1(S)$ with the invariant measure, so $\mu(X) = vol(T_1(S)) = -4\pi^2 \chi(S)$.

As with Basmajian, we use lasers and generate in both directions.

$v \in X = T_1(S)$, define $\gamma(v)$ to be the maximal geodesic

$$\gamma(v) : [T_1, T_2] \to S \quad \text{s.t.} \quad \gamma(v)(t) = \exp(tv). \qquad (T_1 < 0 < T_2)$$

Q: When is $\gamma(v)$ infinite, i.e. $T_1 = -\infty$ or $T_2 = \infty$?

What happens when $\gamma(v)$ is finite (so $\gamma(T_1), \gamma(T_2) \subset \partial S$)?

Let $Z = \{ v \in T_1(S) \mid \gamma(v) \text{ is infinite} \}$. Then by ergodicity of the geodesic flow, $\mu(Z) = 0$!

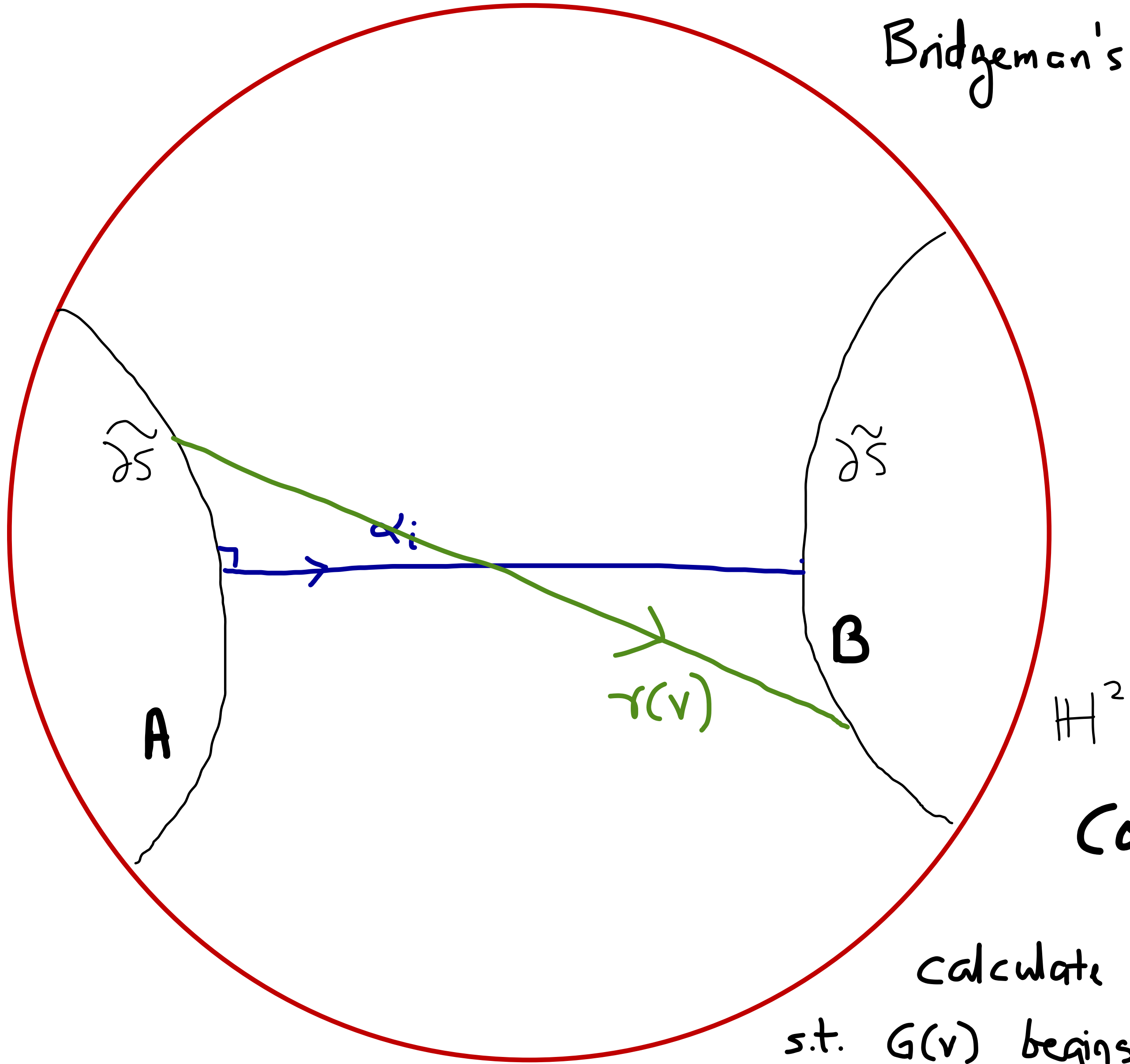
If $v \in T_1(S) \setminus Z$, then $\gamma(v)$ is homotopic rel. to ∂S to a unique oriented orthogeodesic in S, $\gamma(v) \sim \alpha_i$ for some i.

Let $W(\alpha_i) = \{ v \in T_1(S) \mid \gamma(v) \sim \alpha_i \}$.

then $T_1(S) = Z \sqcup \bigsqcup_i W(\alpha_i) \Rightarrow \mu(T_1(S)) = \sum_i \mu(W(\alpha_i))$. $\square$

$= B_r(|\alpha_i|)$.

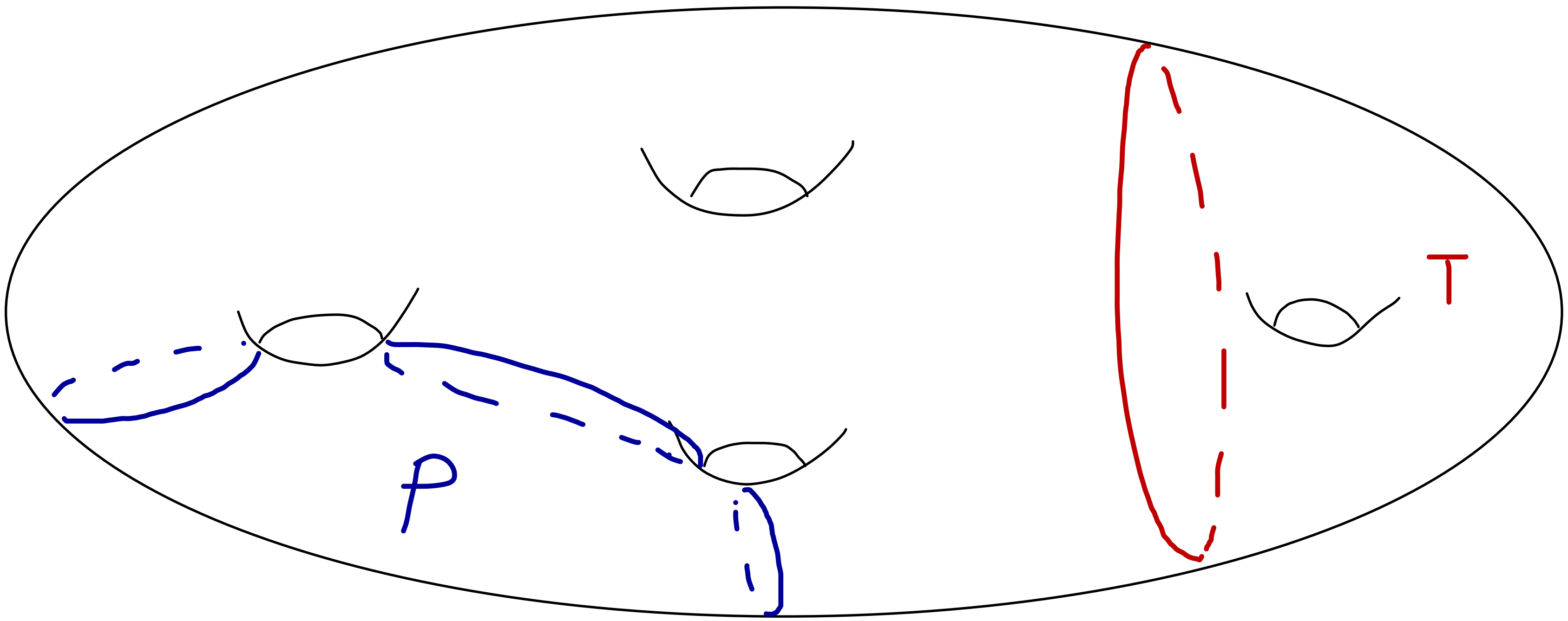
Bridgeman's function:



Computing $\mu(W(\alpha_i))$.

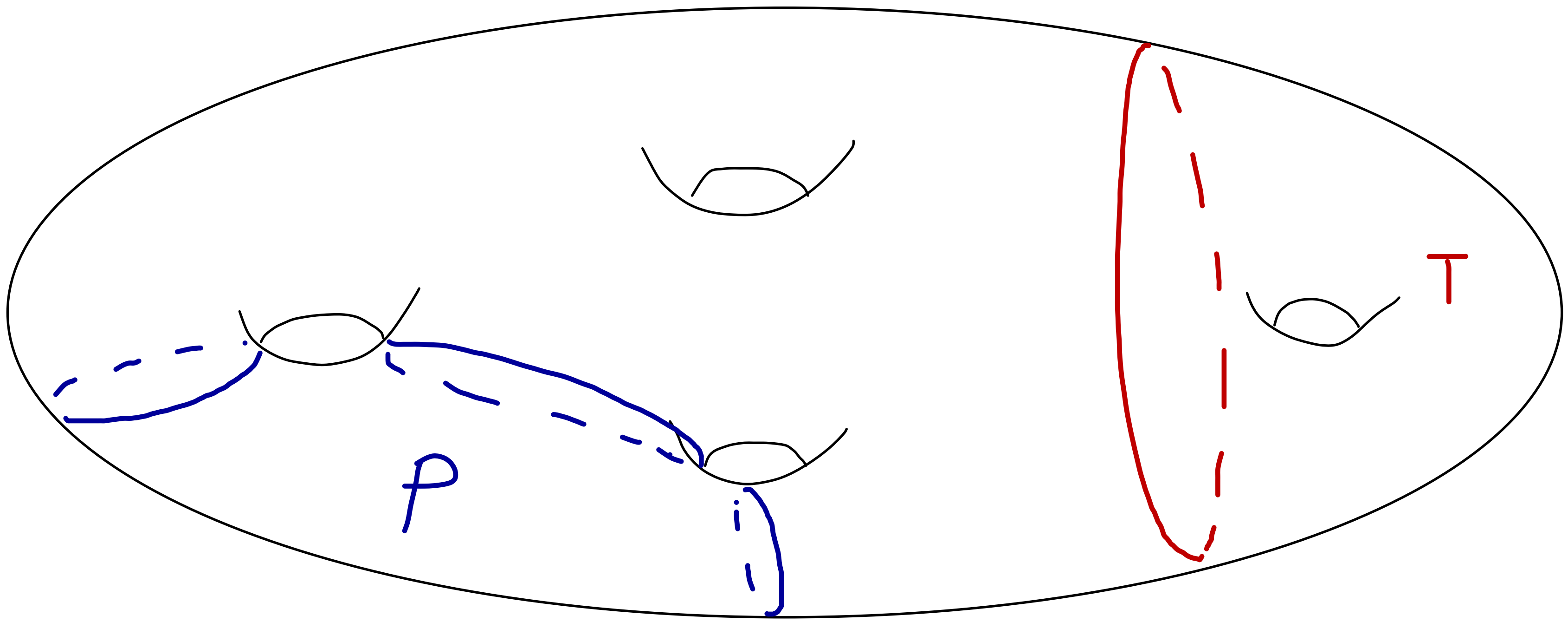
calculate the measure of all $v \in T_1(\mathbb{H}^2)$ s.t. $G(v)$ begins at A & end at B.

3/ <u>Example 4</u> (Identity for closed surfaces, Luo-T).



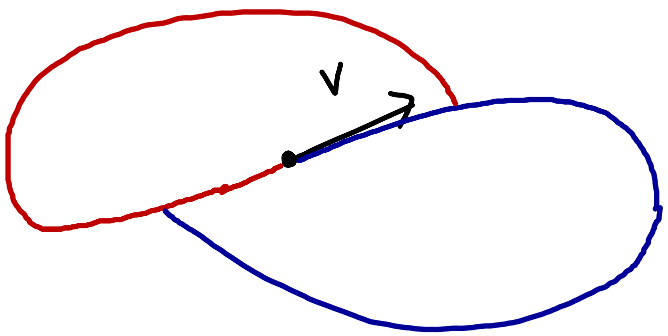
What to do if $\partial S = \phi$?

Key ideas: Start <u>everywhere</u> & every direction & build <u>walls</u>



$X = T_1(S)$, $\mu(X) = -4\pi^2 \chi(S)$.

For $v \in T_1(S)$, $G(v)$ = geodesic obtained by building maximal wall at equal speed in both directions.



$G(v) : [T_1, T_2] \to S$, $G(v)(t) = \exp(tv)$, $G(v)$ is embedding on (T_1, T_2).

- $Z = \{ v \in T_1(S) \mid G(v) \text{ is infinite} \}$ has measure 0 by erg. of geod. flow
- $v \in T_1(S) \setminus Z \Rightarrow G(v)$ is an embedded graph of euler characteristic -1.
- $N(G(v))$ is a surface of euler characteristic -1, i.e. P or T.
- P or T can be made 'geometric' (geodesic boundary) & $v \in T_1(P)$ or $T_1(T)$.

Define $W(P) = \{ v \in T_1(S) \mid G(v) \text{ is a spine for } P \}$
or $W(T)$ for T

Then

$$\mathrm{Vol}(T_1(S)) = -4\pi^2\chi(S) = 0 + \sum_P \mu(W(P)) + \sum_T \mu(W(T)).$$

The functions f & g are $f(P) = \mu(W(P))$, $g(T) = \mu(W(T))$.

This decomposition is more 'complicated' than Exs. 1, 2 & 3.

Computation of f & g are also significantly more difficult — but doable.

$\Rightarrow$ Main Thm.

Ref: A dilogarithm identity on moduli spaces of curves, arxiv 1102.2133 v2
to appear, JDG.

Thank You

спасибо