On Cohen braids

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Plan

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Basic definition

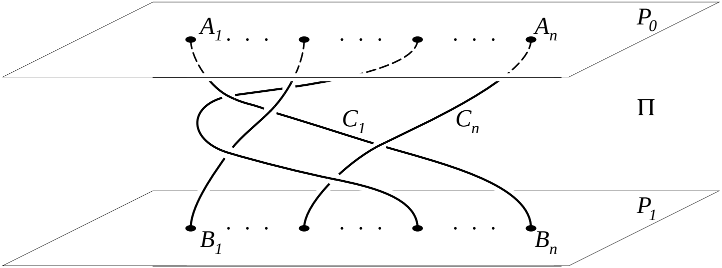
Let M be a general connected surface, possibly with boundary components (we can consider M as a compact surface with a finite number of punctures). Let $B_n(M)$ be the n-strand braid group on a surface M.

For the simplest case of the plane or open (closed) disc braids naturally are considered as objects in 3-space. We remind the corresponding picture. Let us consider two parallel planes P_0 and P_1 in $\mathbb{R}^3$, which contain two ordered sets of points $A_1, \ldots, A_n \in P_0$ and $B_1, \ldots, B_n \in P_1$. These points are lying on parallel lines L_A and L_B respectively. The space between the planes P_0 and P_1 we denote by Π.

Braids

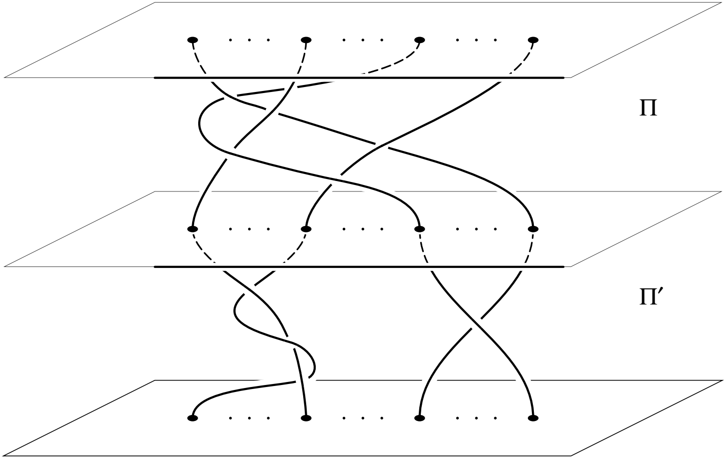
Let us connect the set of points A_1, ..., A_n with the set of points B_1, ..., B_n by simple nonintersecting curves C_1, ..., C_n lying in the space Π and such that each curve meets only once each parallel plane P_t lying in the space Π.
This object is called a *geometric braid* and the curves are called the *strings* of a geometric braid.
Two geometric braids β and β' on n strings are *isotopic* if β can be continuously deformed into β' in the class of braids.
The relation of isotopy is an equivalence relation on the class of geometric braids on n strings. The corresponding equivalence classes are called *braids on n strings*.

Braids



Braids

On the set Br_n of braids the structure of a group introduces as follows.



Braids

Unit element is the equivalence class containing a braid of n parallel intervals, the braid β^{-1} inverse to β is defined by reflection of β with respect to the plane $P_{1/2}$. A string C_i of a braid β connects the point A_i with the pont B_{k_i} defining the permutation S^β. If this permutation is identical then the braid β is called *pure*.
The subgroup of pure braids for a manifold M is usually denoted $P_n(M)$.

Brunnian braids

My talk two years ago at this conference was about *Brunnian braid* on the surface M. Brunnian means a braid that becomes trivial after removing any one of its strands. It is done with the help of operations

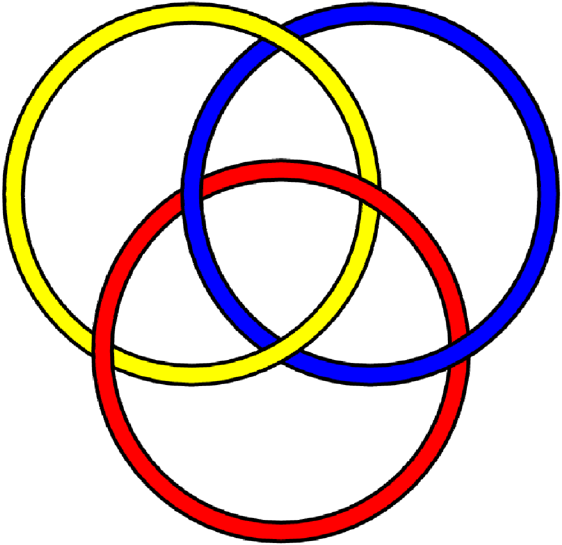
$$d_i\colon B_n(M) \to B_{n-1}(M)$$

which are obtained (roughly speaking) by forgetting the i-th strand, $1 \leq i \leq n$. We can interpret a Brunnian braid $\beta \in B_n(M)$ as a solution of system of n equations

$$\begin{cases} d_1(\beta) = 1, \\ \dots \\ d_n(\beta) = 1. \end{cases}$$

Let $\mathrm{Brun}_n(M)$ denote the set of the n-strand Brunnian braids. Then $\mathrm{Brun}_n(M)$ forms a subgroup of $B_n(M)$.

Brunnian braids



In the present talk we replace the unit element of the group by an arbitrary braid $\alpha \in B_{n-1}(M)$ and we ask the following question: does there exist a braid $\beta \in B_n(M)$ such that it is a solution of the following system of equations

$$\begin{cases} d_1(\beta) = \alpha, \\ \dots \\ d_n(\beta) = \alpha. \end{cases} \tag{1}$$

Apart from Brunnian braids the following example can be given. Let α be the Garside element $\Delta_{n-1} \in B_{n-1}(M)$. Then $\Delta_n \in B_n(M)$ is a solution of system (1).

Equations in braid groups

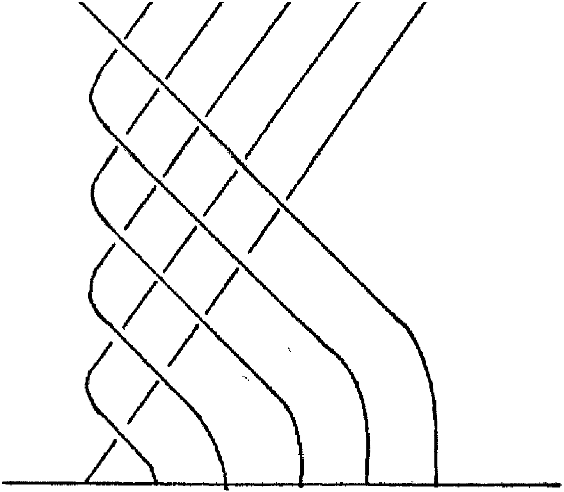


Figure: Garside element Δ_6

Theorem

Let M be any connected 2-manifold such that $M \neq S^2$ or $\mathbb{RP}^2$ and let $\alpha \in B_{n-1}(M)$. Then the equation (1) for n-strand braids β has a solution if and only if α satisfies the condition that

$$d_1\alpha = \ldots = d_{n-1}\alpha.$$

Bi-Δ-structures

The technique of the proof is based on the bi-Δ-group structure on the pure braid groups over connected 2-manifolds with nonempty boundaries as well as the determination of Brunnian braids on general connected 2-manifolds .

Bi-Δ-structures

Let M be a connected manifold with $\partial M \neq \emptyset$. Let a be a point in a collar of ∂M

$$\partial M \times [0,1) \subseteq M.$$

Then the map

$$F(M,n) \simeq F(M \setminus \partial M \times [0,1), n) \to F(M, n+1),$$
$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{i-1}, a, z_{i+1}, \dots, z_n)$$

induces a group homomorphism

$$d^i \colon B_n(M) = \pi_1(F(M,n)/\Sigma_n) \to B_{n+1}(M) = \pi_1(F(M,n+1)/\Sigma_{n+1})$$

for $1 \leq i \leq n+1$.

Bi-Δ-structures

Intuitively, d^i is given by adding a trivial strand in position i. The sequence of groups $\{B_{n+1}(M)\}_{n\geq 0}$ with faces relabeled as $\{d_0, d_1, \ldots\}$ and cofaces relabeled as $\{d^0, d^1, \ldots\}$ forms a bi-Δ-set structure.

The following identities hold:

1. $d_j d_i = d_i d_{j+1}$ for $j \geq i$;
2. $d^j d^i = d^{i+1} d^j$ for $j \leq i$;
3. $d_j d_i = \begin{cases} d^{i-1} d_j & \text{if} \quad j < i, \\ \mathrm{id} & \text{if} \quad j = i, \\ d^i d_{j-1} & \text{if} \quad j > i. \end{cases}$

The sequence of pure braid groups $\{P_{n+1}(M)\}_{n\geq 0}$ is a bi-Δ-group.

Proposition

Let M be a connected 2-manifold with nonempty boundary. Then $P_n(M)$ is the (iterated) semi-direct product of the subgroups

$$d^{i_k} d^{i_{k-1}} \cdots d^{i_1}(\mathrm{Brun}_{n-k}(M)),$$

$1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $0 \leq k \leq n-1$, *with lexicographical order from the right.*

Cohen braids

We return to the case when M is an arbitrary connected 2-manifold. Define a set

$$\mathfrak{H}_n^B(M) = \{\beta \in B_n(M) \mid d_1\beta = d_2\beta = \cdots = d_n\beta\}.$$

Namely $\mathfrak{H}_n^B(M)$ consists of n-strand pure braids such that it stays the same braid after removing any one of its strands. We call this *Cohen braids*. We denote Cohen braids for the disc simply by $\mathfrak{H}_n^B$ as well as Brunnian braids of the disc by Brun_n.
A typical element in $\mathfrak{H}_n^B(M)$ is the half-twist braid

$$\Delta_n = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)\sigma_1.$$

Proposition

Let M be any connected 2-manifold. Then the set $\mathfrak{H}_n^B(M)$ is subgroup of $B_n(M)$. Moreover $d_i(\mathfrak{H}_n^B(M)) \subseteq \mathfrak{H}_{n-1}^B(M)$ and the function

$$d_1 = d_2 = \cdots = d_n \colon \mathfrak{H}_n^B(M) \to \mathfrak{H}_{n-1}^B(M)$$

is a group homomorphism.

Proposition

Let M be any connected 2-manifold. Let $n \geq 2$. Then $\mathfrak{H}_n^B(M) \cap P_n(M)$ is a subgroup of $\mathfrak{H}_n^B(M)$ of index 2.

Let $\mathfrak{H}_n(M) = \mathfrak{H}_n^B(M) \cap P_n(M)$. Then $d_1(\mathfrak{H}_n(M)) \subseteq \mathfrak{H}_{n-1}(M)$. This gives a tower of groups

$$\cdots \to^{d_1} \mathfrak{H}_n(M) \to^{d_1} \mathfrak{H}_{n-1}(M) \to^{d_1} \cdots .$$

Let $\mathfrak{H}(M) = \lim\limits_{n} \mathfrak{H}_n(M)$ be the inverse limit of the tower of groups.

Proposition

Let M be any connected 2-manifold such that $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$. Then

$$d_1 : \mathfrak{H}_n(M) \to \mathfrak{H}_{n-1}(M)$$

is an epimorphism for each $n \geq 2$.

Corollary

Let M be any connected 2-manifold such that $M \neq S^2$ or $\mathbb{RP}^2$. Then there exists the following short exact sequence

$$1 \to \mathrm{Brun}_n(M) \to \mathfrak{H}_n(M) \xrightarrow{d_1} \mathfrak{H}_{n-1}(M) \to 1 \tag{2}$$

which connects the nth $(n-1)$th Cohen braid groups and Brunnian braids on n-strands.

In small dimensions we have $\mathfrak{H}_2 = \mathrm{Brun}_2 = P_2 = \mathbb{Z}$ and the exact sequence (2) gives

$$1 \to \mathrm{Brun}_3 \to \mathfrak{H}_3 \xrightarrow{d_1} \mathbb{Z} \to 1$$

In $\mathfrak{H}_3$ it is the central element Δ_3^2 which is mapped to the generator of $\mathfrak{H}_2$, so $\mathfrak{H}_3 \cong \mathbb{Z} \times \mathrm{Brun}_3$ and for the next step we have

$$1 \to \mathrm{Brun}_4 \to \mathfrak{H}_4 \longrightarrow \mathbb{Z} \times \mathrm{Brun}_3 \to 1.$$

Question:
What can one say about Cohen braids on the sphere and projective plane?

In dimension 3 we have $\mathfrak{H}_3(S^2) = \mathrm{Brun}_3(S^2) = P_3(S^2) = \mathbb{Z}/2$, the map $\mathfrak{H}_4(S^2) \xrightarrow{d_1} \mathfrak{H}_3(S^2)$ is obviously onto and Δ_3^2 is central in $\mathfrak{H}_4(S^2)$, so we have

$$\mathfrak{H}_4(S^2) \cong \mathbb{Z}/2 \times \mathrm{Brun}_4(S^2).$$

Calculations show that

$$\mathfrak{H}_2(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/4.$$