KHOVANOV HOMOLOGY OF KNOTS IN A THICKENED TORUS

A.A. Akimova

South Ural State University (SUSU)

Laboratory of Quantum Topology, Chelyabinsk State University(CSU)

International Conference "Geometry days in Novosibirsk, 2014"

Introduction

In 2000 M. Khovanov described construction of a special graded chain complex associated with a given diagram of an oriented link L in the 3-sphere and proved

Proposition

For an oriented link L,

$$K(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j dim_{\mathbb{Q}}(\mathcal{H}^{i,j}(L) \otimes \mathbb{Q})$$

where $K(L)$ is the Kauffman bracket.

We consider a natural generalization of this construction in the case of knots in a thickened torus $T \times I$.

Such generalization for knots with additional structure was proposed by V.O. Manturov in 2007.

Problem statement

To calculate Khovanov homology of knots in $T \times I$ having diagrams with at most 3 crossings.

The table of such knots and the list of their generalized Kauffman polynomials were composed by S.V. Matveev and A.A. Akimova in 2012.

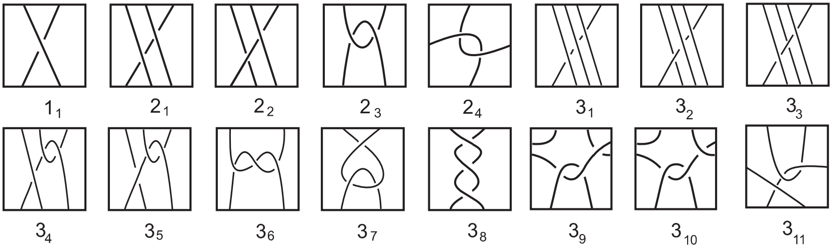


Figure : Knot diagrams in T with at most 3 crossings

Generalized Kauffman polynomial

$$X(K) = (-a)^{-3\omega(K)} \sum_{s} a^{\alpha(s)-\beta(s)} (-a^2 - a^{-2})^{\gamma(s)} x^{\delta(s)}$$

- State s is obtained by resolving all crossing points
- $\alpha(s)$, $\beta(s)$ are the numbers of types A and B of smoothing in a given state s
- $\gamma(s)$, $\delta(s)$ are the numbers of trivial and nontrivial circles in T in a given state s
- $\omega(K)$ is the writhe of the diagram
- The sum is taking over all states

Figure : Markers

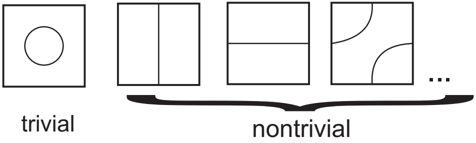
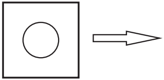
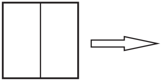


Figure : Trivial and nontrivial circles in T

Chain complex of graded vector spaces: groups



trivial



nontrivial

Vector spaces:

$$V = \{1, X\}$$

$$W = \{\dot{1}, \dot{X}\}$$

$s \Rightarrow$ a vector space which is a tensor product of $\gamma(s)$ spaces of type V and $\delta(s)$ spaces of type W

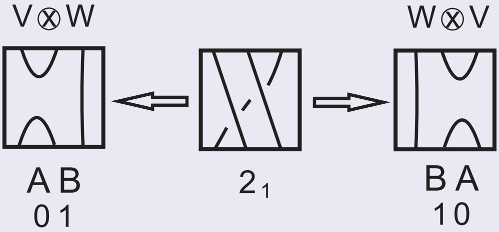
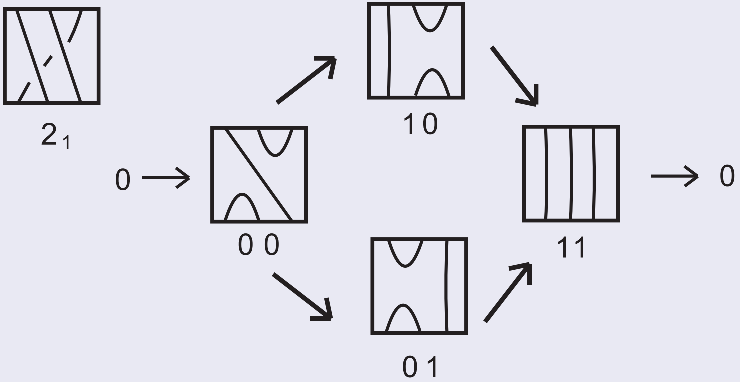


Figure : An example: we associate state s with a vector space

An example of a calculation



$$0 \mapsto W \mapsto W \otimes V \oplus V \otimes W \mapsto W \otimes W \otimes W \mapsto 0$$

Chain complex of graded vector spaces: differentials

We consider 9 linear maps:

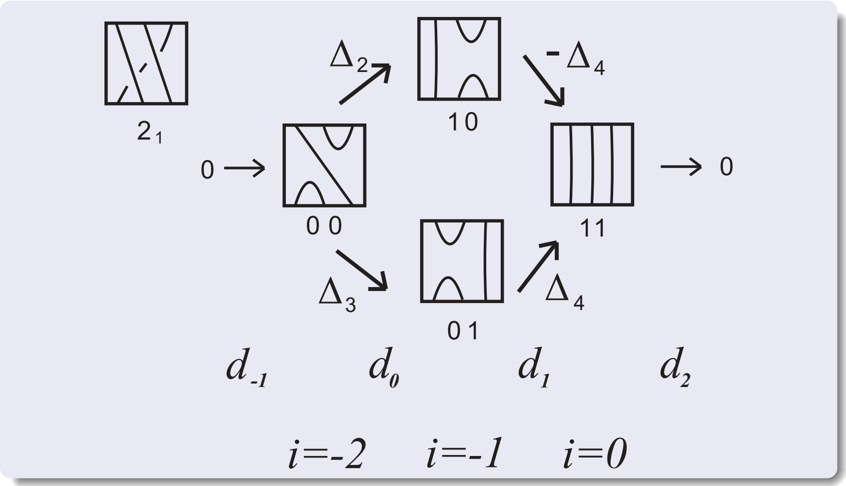
$$\eta : W \mapsto W$$

$$m_1 : V \otimes V \mapsto V \qquad \Delta_1 : V \mapsto V \otimes V$$

$$m_2 : W \otimes V \mapsto W \qquad \Delta_2 : W \mapsto W \otimes V$$

$$m_3 : V \otimes W \mapsto W \qquad \Delta_3 : W \mapsto V \otimes W$$

$$m_4 : W \otimes W \mapsto V \qquad \Delta_4 : V \mapsto W \otimes W$$



$d_0 = \Delta_2 \oplus \Delta_3$ $d_1 = (-\Delta_4, \Delta_4)$

Chain complex of graded vector spaces

The constructed vector space has 3 gradings:

homological: $i(s) = \frac{\omega(D)-\sigma(s)}{2}$

quantum: $j(s) = \frac{3\omega(D)-\sigma(s)-2\tau(s)+2\phi(s)}{2}$

additional ("dotted"): $k(s) = \phi(s)$

Where

$\sigma(s) = \alpha(s) - \beta(s)$

$\tau(s) = \natural 1(s) - \natural X(s)$

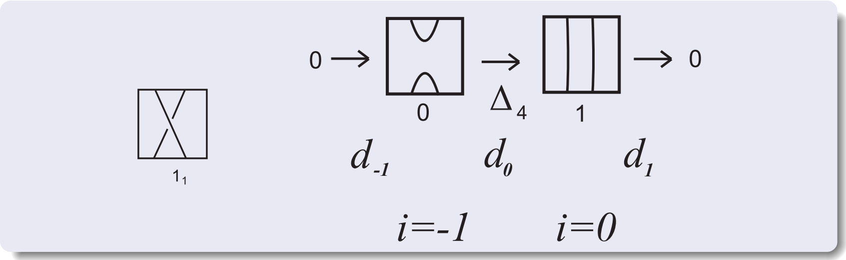
$\phi(s) = \natural \dot{X}(s) - \natural \dot{1}(s)$

Khovanov polynomial is the following:

$$Kh(K) = \sum_s t^{i(s)} q^{j(s)} g^{k(s)}$$

$$Kh(K)\mid_{t=-1} = \sum_{i=s} (-1)^{i(s)} q^{j(s)} g^{k(s)} = X(K),$$

where $X(K)$ is the generalized Kauffman polynomial with change of variables: $a = (-q)^{-1/2}$ and $x = qg^{-1} + q^{-1}g$.



$$\mathcal{H}^0 = ker(d_0)/im(d_{-1}) =$$
$$=< X > / < 0 >=< X >$$

$$\mathcal{H}^1 = ker(d_1)/im(d_0) =$$
$$=< \dot{1}\otimes\dot{1}, \dot{1}\otimes\dot{X}, \dot{X}\otimes\dot{1}, \dot{X}\otimes\dot{X} >$$
$$/ < \dot{1}\otimes\dot{X} + \dot{X}\otimes\dot{1} >=$$
$$=< \dot{1}\otimes\dot{1}, \dot{X}\otimes\dot{1}, \dot{X}\otimes\dot{X} >$$

$\mathbf{1_1}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
i	-1	0	0	0
j	-3	1	-1	-3
k	0	-2	0	2

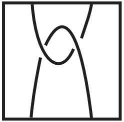
Khovanov homology of knots $2_1 - 2_3$

2_1

$\mathbf{2_1}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
i	-1	-1	0	0	0	0
j	-3	-5	1	-1	-3	-5
k	-1	1	-3	-1	1	3

2_2

$\mathbf{2_2}$	$\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
i	-1	-1	0	0	0	0	1	1
j	-1	-3	1	3	-1	-3	3	1
k	-1	1	-1	-3	1	3	-1	1



2_3

$\mathbf{2_3}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$
i	-2	-1	-1	-1	0	0	0
j	-5	-1	-5	-3	1	-1	-3
k	0	-2	2	0	-2	0	2

Thank you for your attention!