Almost Contacts Structures on Five-dimensional Manyfollds

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The concept of Almost Contact Structure is generalization of Contact Structure for 1-form on odd-dimensional manyfolds, when 1-form has a arbitrary radical. As known, if α is an Contact Form then α satisfies the condition:

$$d\alpha^n \wedge \alpha \neq 0,$$

and radical of 1-form α is the one-dimensional line transversal to the contact distribution. If ξ is the vector field that generates this line then $\operatorname{rad} \alpha = \mathbb{R} \otimes \xi$. For Almost Contact Form the condition $d\alpha^n \wedge \neq 0$ is not necessary and radical may have an arbitrary dimension.

Let M be a smooth real manyfold with dimension $2n+1$ and α be a smooth 1-form on M. The radical of 1-form α is a vector fields variety

$$\operatorname{rad}\alpha = \{X \in C^1(TM) : d\alpha(X,Y) = 0 \quad \forall Y\}.$$

If vector subbundle $\operatorname{rad}\alpha$ has a constant rank over M then α is called a regular 1-form. If $\operatorname{rad}\alpha = TM$ then α is a closed 1-form. By Eugene Kornev in [2] has been proved the next result:

Theorem 1.

Let α be a regular unclosed 1-form on smooth manyfold M with dimension $n \geq 3$.
(1) *If n is even then $\operatorname{rad}\alpha$ has a even rank and*

$$0 \leq \dim(\operatorname{rad}\alpha) \leq n-2.$$

(2) *If n is odd then $\operatorname{rad}\alpha$ has odd rank and*

$$1 \leq \dim(\operatorname{rad}\alpha) \leq n-2.$$

Let g be a Riemannian metric on M.

Definition 2.

Almost Contact Structure on odd-dimensional manyfold M is a pair (α, ξ), where α is a regular 1-form on M, ξ is a vector field on M so that $\alpha(X) = g(\xi, X) \quad \forall X \in C^1(TM)$. ξ is called a characteristic vector field. The triple (α, ξ, g) is called a almost contact metric structure.

The important class of almost contact structures (α, ξ) is a case when $\xi \in \operatorname{rad} \alpha$. These almost contact structures are called strictly almost contact structures. Homogeneous strictly almost contact structures on three-dimensional manyfolds have been classified by G. Calvaruso in [1]. By this matter, we consider almost contact structure on five-dimensional manyfolds. The theorem 1 follows that on a five-dimensional manyfold any regular 1-form may have only radical of dimension 1, 3, or 5. If $\dim(\operatorname{rad} \alpha) = 5$ then α is a closed 1-form. If $\dim(\operatorname{rad} \alpha) = 1$ and $\xi \in \operatorname{rad} \alpha$ then α is a classic contact form. Now, we provide the example of almost contact form with radical of dimension 3.

Let M be a smooth five-dimensional manyfold, f and h be a smooth function om M so that 1-forms df and dh are linear independent at each point. Then, 1-form $\alpha = f dh$ is a regular 1-form, cause $d\alpha = df \wedge dh$. We have that

$$\operatorname{rad} \alpha = \ker df \cap \ker dh.$$

Since $\dim(\ker df \cap \ker dh) = 3$ we obtain that $\dim(\operatorname{rad} \alpha) = 3$. Now, we provide example of manyfold that admits almost contact structure, but no admits classic contact structures.

Let Q be a $2n$-dimensional smooth Riemannian manyfold with Riemannian metric g_0, f be a smooth function on Q totaly no vanishin over Q, $M = Q \rtimes \mathbb{R}$, and $\xi = \frac{d}{dt}$ be a basic vector field on $\mathbb{R}$. We can construct a Riemannian metric on M as $g = g_0 + f^2 dt^2$ and consider that $[X, \xi] = df(X)\xi \quad \forall X \in C^1(TQ)$, where $[X, Y]$ is Lie bracket of vector fields X, Y. Let us consider that α is 1-form on M so that

$$\alpha(X) = g(\xi, X) \quad \forall X \in C^1(TM).$$

It is easy to see that 1-form α is almost contact form, rather than contact form, cause $\ker \alpha = TQ$ is involutive distribution. For any $X \in C^1(TQ)$

$$d\alpha(X, \xi) = \frac{d(\alpha(\xi))(X) - df(X)\alpha(\xi)}{2}.$$

Since $d\alpha(X, \xi)$ is a 1-form on Q and $d\alpha|_Q \equiv 0$ we obtain that $\dim(\operatorname{rad} \alpha) = 2n - 1$.
More over, any 1-form η so that $\eta(\xi) = 0$ should not be a contact form, cause it satisfies the condition

The important class of almost contact structure is a case when characteristic vector field ξ has a constant length. Intersection of this class and class of strictly almost contact structures is described by next theorem:

Theorem 3.

Let (α, ξ, g) be a almost contact metric structure on smooth $2n+1$-dimensional manyfold M and $g(\xi, \xi) = \mathrm{const}$ then ξ belongs to $\operatorname{rad} \alpha$ if and only if:
(1) *ξ is a geodesic vector field, i. e. $\nabla_\xi \xi = 0$.*
(2) *$L_\xi \alpha = 0$, where L_ξ is a Lie derivation along with ξ.*
(3) *For any vector field X on M $[X, \xi]$ is orthogonal to ξ at each point.*

In the current time no exists a total classification of homogeneous strictly almost contact structures for five-dimensional spaces. But we can obtain some particular results.

It is known that five-dimensional sphere S^5 can be viewed as homogeneous space $\mathrm{SO}(6)/\mathrm{SO}(5)$. By Eugene Kornev has been proved the next result:

Theorem 4.

For any $n \in \natural$ on sphere S^{2n+1} no exist $\mathrm{SO}(2n+1)$-invariant unclosed almost contact structures.

This theorem follows that five-dimensional sphere S^5 no admits SO(5)-invariant almost contact structures with radical of dimension 1 and 3. However, when group G acts on S^5 nontransitively we can provide a G-invariant almost contact structure.

Consider a Hopf Bundle $S^5 \to \mathbb{C}p^2$ with fibre $S^1 \cong \mathrm{U}(1)$. Let Q be a connection on S^5, ω be a connection form, and Ω be a connection curvature form. Both ω and Ω are S^1-invariant forms. The structure equation follows that $d\omega = \Omega$. Let us consider that ξ is vector field tangent to orbit of S^1 action on sphere S^5 so that $\omega(X) = g(\xi, X) \quad \forall X \in C^1(TS^5)$, where g is a metric of embedding $S^5 \to \mathbb{C}^3$. If Q is a flat connection then ω be a closed form. Otherwise, (ω, ξ) be a S^1-imvariant almost contact structure with radical of dimension 1 or 3.

Let G be a five-dimensional unsolvable Lie Group and $\mathfrak{g}$ be its Lie Algebra. The Levi-Maltsev theorem follows that $\mathfrak{g} \cong \mathfrak{s} \rtimes \mathfrak{r}$, where $\mathfrak{r}$ is isomorphic to whether $\mathbb{R}^2$, or $\mathfrak{e}(1)$ (Lie algebra of real line affine transformations group E(1)) and $\mathfrak{s}$ is isomorphic to whether $\mathfrak{so}(3)$, or $\mathfrak{sl}(2, \mathbb{R})$. Theorem 1 follows that radical of any left-invariant almost comtact structure on G may have only dimension 1,3, or 5. For Lie Groups space of left-invariant 1-forms having a radical of maximal dimension (closed 1-forms) dimension is the first Betti Number. For five-dimensional unsolvable Lie Groups we have the next result:

Theorem 5.

Let G be a five-dimensional unsolvable Lie Group. Then space of left-invariant 1-form having a radical of dimension 5 (closed 1-forms) may have onli dimension 0,1, or 2.

Let α be a regular 1-form with three-dimensional radical on unsolvable five-dimensional Lie Group $g : \mathfrak{g} \cong \mathfrak{s} \rtimes \mathfrak{r}$. We define a radical index concept as $\dim(\mathfrak{s} \cap \operatorname{rad} \alpha)$. A radical index may possess a value only 1,2, or 3.

Some results for five-dimensional unsolvable Lie Groups are collected in the following theorem:

Theorem 6.

Let $G : \mathfrak{g} \cong \mathfrak{s} \rtimes r$ (see the slide 10) be a five-dimensional Lie Group. Then take place the next statements:
(1) *If $[\mathfrak{s}, \mathfrak{r}] = 0$ and $\mathfrak{r} = \mathbb{R}^2$ then any unclosed left-invariant almost contact structure has radical with dimension 3 and index 1.*
(2) *If $[\mathfrak{s}, \mathfrak{r}]$ is a real line in $\mathfrak{r}$ and $\mathfrak{r} \cong e(1)$ then G admits the left-invariant almost contact structure having radical with dimension 3 and index 3 (radical is $\mathfrak{s}$).*
(3) *If $[\mathfrak{s}, \mathfrak{r}] = \mathfrak{r}$ then G no admits left-invariant almost contact structures having radical with dimension 3 and index 3.*
(4) *If (α, ξ) is a unclosed left-invariant almost contact structure on G and $\mathfrak{r} \subset \ker \alpha$ then α has radical with dimension 3 and index 1.*
(5) *If $\mathfrak{s} \cong \mathfrak{so}(3)$ then G no admits left-invariant almost contact structures having radical with dimension 3 and index 2.*

As example of nilpotent group we consider the five-dimensional Heisenberg Group H_5. The Lie Algebra $\mathfrak{h}_5$ of Heisenberg Group H_5 admits the natural left-invariant basis $e_1, \dots, e_5$ with nonzero commutators

$$[e_2, e_1] = e_5, \ [e_4, e_3] = e_5,$$

and dual basis of left-invariant 1-forms $e_1^*, \dots, e_5^*$ so that

$$de_1^* = de_2^* = de_3^* = de_4^* = 0,$$
$$de_5^* = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*.$$

By this way, we obtain that the space of left-invariant almost contact structures having radical with dimension 5 on H_5 has dimension 4, the space of left-invariant almost contact structures having radical with dimension 1 has dimension 1, and H_5 no admits left-invariant almost contact structures having radical with dimension 3.

- G. Calvaruso, “Three-dimensional homogeneous almost contact metric structures.”, *Journal of Geometry and Physics*, Vol. 69, 60-73 (2013).
- E. Kornev, “Invariant Affinor Metric Structures on Lie Groups.”, *Siberian Mathematical Journal*, Vol. 53, No. 1, 89-102 (2012).