Infinitesimal deformations of rotational surfaces with flattening at poles

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Conference in honour of 85-anniversary of

Yu.G. Reshetnyak

Novosibirsk, 27.09.2014

Plan of the talk

1. A bit of history
2. Equations
3. Corrugated (or goffered) surfaces of revolution
4. Local inf. flexibility near the pole
5. Surfaces with two poles
6. Theorem of existence
7. Local 2nd order inf. deformations
8. Global 2nd order inf. deformations

Known results

Cohn-Vossen (1929) An example of infinitesimally flexible (i.f.) rotational surface with one harmonic.

Reshetnyak (1962) C^∞-smooth i.f. rotational surface with exactly any a priori given numbers $2 \leq n_1 \leq n_2 \leq ... \leq n_k < \infty$ of harmonics.

Trotsenko (1980) The same result in the analytic case.

Efimov (1948) Existence of locally infinitesimally rigid (i.r.) surfaces in the analytical class of surfaces and deformations.

S. (1969) $C^n(1 \leq n \leq \infty)$-smooth non-convex surfaces of revolution locally and globally i.r. in the class of C^1-smooth deformations.

Efimov and Usmanov (1973) A class of convex rotational surfaces locally i.r. in the class of C^∞-smooth deformations.

S. (1986) Some criteria for i.r. of compact rotational surfaces with flattening at poles

The case of inf. deformations of 2nd order

Cohn-Vossen (1929)A criterion for infinitesimal flexibility of second order for compact rotational surfaces.

Poznyak (1961) Existence of a 2nd order i.f. rotational surface.

Ivanova-Karatopraklieva and S. (1989) Local 2nd order i.r. and i.f. of a rotational surface at pole with flattening in different classes of smoothness.

I'll speak on some results mentioned above as well as on ones published in my article
Жесткость и неизгибаемость "в малом"и "в целом"поверхностей вращения с уплощениями в полюсах// Математический сборник (2013), т. 204:10, с. 127-160 (Infinitesimal and global rigidity of surfaces of revolution with flattening at poles // Sbornik: Mathematics (2013), v. 204:10, p.) and in the article Бесконечно малые изгибания 2-го порядка поверхностей вращения с уплощением в полюсах (addmited in Математический сборник)

A surface and its infin. deformation of 1st order

A meridional curve is $\mathbf{z} = \varphi(\rho) \in \mathbf{C^n}, \mathbf{1} \leq \mathbf{n} \leq \infty$ or $\varphi(\rho)$ is analytic, $0 \leq a \leq \rho \leq b$, so S - the surface of revolution around the axis Oz has the equation

$$S : z = \varphi(\sqrt{x^2 + y^2}).$$

For the analytic case one should be

$$\varphi(\sqrt{x^2 + y^2}) = \rho^{2k} \sum_{n=0}^{\infty} a_n \rho^{2n}, \rho^2 = x^2 + y^2, a_0 \neq 0, k \geq 1.$$

In a vectorial form

$$S : \mathbf{r} = \rho\mathbf{e}(\theta) + \varphi(\rho)\mathbf{k}, \mathbf{0} \leq \theta \leq \mathbf{2}\pi,$$

where the vector $\mathbf{e}(\theta)$ describes the unit circle.
A field of infinitesimal deformation U is searched in the form

$$\mathbf{U} = \alpha(\rho, \theta)\mathbf{k} + \beta(\rho, \theta)\mathbf{e} + \gamma(\rho, \theta)\mathbf{e}'(\theta). \tag{1}$$

By the definition for the metric ds_t^2 of the deformed surface $S_t : \mathbf{r_t} = \mathbf{r} + \mathbf{tU}$ should satisfy to the relation

$$ds_t^2 - ds^2 = o(t), t \to 0,$$

so for the vector field $\mathbf{U}$ we have an equation

$$d\mathbf{r}d\mathbf{U} = \mathbf{0}. \tag{2}$$

Presenting the coefficients α, β and γ from (1) by their Fourier expansions

$$\alpha = \sum_{-\infty}^{\infty} \alpha_m(\rho) e^{im\theta}, \beta = \sum_{-\infty}^{\infty} \beta_m(\rho) e^{im\theta}, \gamma = \sum_{-\infty}^{\infty} \gamma_m(\rho) e^{im\theta}$$

(where $\alpha_{-m} = \bar{\alpha_m}, \beta_{-m} = \bar{\beta_m}, \gamma_{-m} = \bar{\gamma_m}$) and using the equation (2) we obtain a system of differential equations

$$\alpha_m' - \frac{m^2}{\rho}\alpha_m - \frac{m^2-1}{\rho\varphi'}\beta_m = 0 \tag{3}$$

$$\beta_m' + \frac{m^2\varphi'}{\rho}\alpha_m + \frac{m^2-1}{\rho}\beta_m = 0, m \geq 2 \tag{4}$$

(and $\gamma_m(\rho) = \frac{i}{m}\beta_m(\rho)$). The functions $\alpha_m(\rho), \beta_m(\rho), \gamma_m(\rho)$ compose (and often are called) m-th harmonic of the field $\mathbf{U_1}$.

At the pole (where $\rho = 0$ and $\varphi(0) = \varphi'(0) = 0$) one should be

$$\alpha_m(0) = \alpha'_m(0) = \beta_m(0) = \beta'_m(0) = 0. \tag{5}$$

If the surface $S \in C^2$ and the field of i.d. is in C^2 too, the system (3)-(4) can be reduced to an equation

$$\rho\varphi'(\rho)\alpha''(\rho) + \rho\varphi''(\rho)\alpha'(\rho) - m^2\varphi''(\rho)\alpha(\rho) = 0. \tag{6}$$

So we have to study solutions of the system (3)-(4) or of the equation (6) with the initial conditions (5).

Equations for the inf. deformations of 2nd order

A 2nd order inf. deformation is presented as follows

$$S_t : \mathbf{r_t} = \mathbf{r} + \mathbf{2tU_1} + \mathbf{2t^2U_2} \tag{7}$$

with the condition

$$ds_t^2 - ds^2 = o(t^2), t \to 0$$

which gives a system of equations

$$d\mathbf{r}d\mathbf{U_1} = \mathbf{0},\ \mathbf{drdU_2} + (\mathbf{dU_1})^\mathbf{2} = \mathbf{0}. \tag{8}$$

Thus the part $\mathbf{U_1}$ of the deformation (7) presents a field of inf.def. of 1st order, and by this reason if for a field of inf. def. of 1st order $\mathbf{U_1}$ there exists a field $\mathbf{U_2}$ satisfying the second equation of the system (8) in this case one says that the 1st order field $\mathbf{U_1}$ admits an extension to the field of inf. deformation of the 2nd order. As to the equations for the 2nd order inf. deformations we'll discuss them later.

In the classical case considered by Cohn-Vossen it is supposed that in a neighbourhood of the pole there is no other singularity except the pole itself. But the equations (3)-(4) have singularities at points where $\varphi'(\rho) = 0$ too. Suppose that $\rho \in (a, b), 0 \leq a < b < \infty$ and that the zeros of $\varphi'(\rho)$ compose in (a, b) a discrete countable set A and that $|\varphi'(\rho)|$ is piece-wise monotone and has only one local maximum between two successive zeros of $\varphi'(\rho)$. Moreover, one of points $\rho = a$ or $\rho = b$ or both of them are limit points of A.

1) The point $\rho = b$ is a limit point of the set A. Then the surface $S : z = \varphi(\sqrt{x^2 + y^2}), a < \rho^2 = x^2 + y^2 < b^2$ is inf.rigid.

2) The point $a > 0$ is a limit point of A. Then the surface S is inf.rigid.

3) The point $a = 0$ is a limit point of A with the condition $\frac{\rho_n}{\rho_{n+1}} \to 1, n \to \infty$, where the points $\rho_1 > \rho_2 > ...\rho_n > \rho_{n+1} > ...$ are zeros of $\varphi'(\rho)$ and $\rho_n \to a = 0$. Then surface S is inf.rigid.

We would like to underline that:
1) here the infinitesimal rigidity is established for deformations with only C^1-smoothness and without any restriction to the behavior of the field of inf. def. on the boundary;
2) the open surface S can be even analytic and bounded as well as no bounded. If the surface S is compact (that is the pole $\rho = 0$ and the boundary on $\rho = b$ are included in S) then it can be of any smoothness $C^n, 1 \leq n \leq \infty$.

A remark:

If in the case 3) one has $\frac{\rho_n}{\rho_{n+1}} \to c > 1, n \to \infty$, then the surface can be inf. flexible.

A conjecture:

Bendings seeming to Nash-Kuiper bendings in the class of C^1-smooth surfaces are impossible in the class of deformations which were C^1-smooth relatively to the parameter of deformation.

Now we consider the case when $\varphi(0) = 0$ and $\varphi'(\rho) > 0$ in a vicinity of the pole, say, in an interval $(0, \rho_0)$. If, in addition, $\varphi(\rho) \in C^2$ and $\varphi'(0) > 0$ then we have the classical case.

But under the condition C^1-smoothness of a considered surface as well as C^1-smoothness of deformations there exists no any result. So we begin by establishing some theorems about the uniqueness and existence of inf. flexibility of the surface. These theorems don't have a short formulation and one read it in the abstract of my talk. For example, it is shown that any convex C^1-smooth surface of revolution is inf. flexible in the class of C^1-smooth deformations.

In order to see that this result is not evident it is enough to remark that such an property is not valid in the class of C^2-smoothness: there exits a convex C^2-smooth surface of revolution which near the pole is inf. rigid in the class of C^2-smooth deformations.

The case of analytical smoothness is more interesting. In this case we can work with the second-order equation

$$\rho\varphi'(\rho)\alpha''_m(\rho) + \rho\varphi''(\rho)\alpha'_m(\rho) - m^2\varphi''(\rho)\alpha_m(\rho) = 0. \qquad (6)$$

Using the Fuchs theory we find a solution of the equation (6)

$$\alpha_m(\rho) = \rho^{\nu_m}\sum_{n=0}^{\infty} A_n\rho^{2n}, A_0 \neq 0,$$

where

$$\nu_m = 1 - k + \sqrt{(2k-1)m^2 + (k-1)^2}, \textit{and } \varphi(\rho) \sim C\rho^{2k}, \rho \to 0.$$

The component $\alpha_m(\rho)e^{im\theta}\mathbf{k}$ of the searched field of inf. deformation can be analytical only under the condition

$$\nu_m = m + 2p$$

and it impose to the value of m a condition: the number

$$\sqrt{(2k-1)m^2 + (k-1)^2} + 1 - k - m$$

should be an even number, the natural number k being given. It turns out that this condition is fulfilled only for some special values of m.

Namely the number m of a harmonic should be found by the solutions of the following Diophantus equation

$$X^2 - (2N+1)Y^2 = N^2, \qquad (9)$$

known as the Pell's equation; here

$$X = \nu_m + k - 1, Y = m, \; N = k - 1.$$

(we recall that the meridian $\varphi(\rho) \sim C\rho^{2k}, \rho \to 0$). In the classical case when the pole is not a point of flatness we have $k = 1$ and the equation (9) gives always the value $\nu_m = m + 2p$ with $p = 0$.
The theory of Pell's equation is well developed. In the case $N = k - 1 = 4$ there is no any solution. In the ca $2N + 1 \neq (2a+1)^2$ the number of solutions of the equation (9) is infinite but the numbers of nontrivial harmonics go with big lacunae (see the table 1).

In the cases $2N + 1 = (2a + 1)^2$ there are only a finite number of solutions (we call this case "square order of flattening") but the quantity of solutions is very small – among the first 1000 surfaces with square order of flattening there are only 2 surfaces with 6 harmonics and 1 surface with 5 harmonics (see tables 2 and 3). As a result we can say that the above mentioned theorems by Reshetnyak and Trotsenko are not valid even locally for surfaces with a flatness at pole.

Let's consider now the surfaces of revolution homeomorphic to sphere. They have two poles. Suppose both of poles are points with different order of flatness $2k_1 - 2 = 2N, 2k_2 - 2 = 2M$. Then for m, the number of harmonic, we have a system of two Pell equations:

$$X^2 - (2N+1)Y^2 = N^2 \tag{10}$$

$$Z^2 - (2M+1)Y^2 = M^2 \tag{11}$$

where $Y = m$. If the orders of flatness at both poles are such that the system (10)-(11) has a solution we'll say that these orders of flatness are compatible. If the orders of flatness at poles are not compatible then this surface is infinitesimally rigid.

The system (10)-(11) can be reduced to one Diophantus equation of degree 4 for two unknowns. Such equations can have only a finite number of solutions. In the case of square order of flatness at both poles it is easy to verify whether the orders are compatible and to find Y that is the existing numbers of harmonics. It turns out that the cases of compatibility of orders are rare (see the table 4).

In the general case a nontrivial algorithm for solution of the system (10)-(11) is found by A.Yu. Nesterenko. At the table 5 one can see all compatible pairs for the first 100 orders of flatness.

Remark. The compatibility of orders of flatness at poles is only a necessary condition of inf. bendability of the considered surface of revolution but it is not sufficient for its inf. bendability.

Given a natural number $m > 2$ and a pair of (not necessarily integer) numbers $k_1 \geq 1, k_2 \geq > 1$, there exists a surface of revolution S with flattening of orders $p = 2k_1 - 2 \geq 0$ and $q = 2k_2 - 2 \geq 0$ at the poles such that it admits infinitesimal bendings with a nontrivial m-th harmonic. Furthermore, any of the following combination of flattening is possible:
1) the order of flattening at both poles is $p = q = 0$ (that is actually there is no flattening);
2) there is a flattening at one pole and no flattening at the other pole;
3) both poles have flattening.

In all cases both the surface and the field of infinitesimal bending can be assumed to be analytic away from the poles, and the smoothness at the poles is as follows:
1) both the surface and the infinitesimal bending are analytic if actually there is no flattening at the poles;
2) both the surface and the infinitesimal bending are analytic everywhere if the orders of flattening at both poles are even and compatible for the given number m of harmonic;
3) in all other cases the class smoothness $C^n, n \geq 2,$ of the surface and the field of infinitesimal bending are defined by the values of m, k_1, k_2.

Now we consider C^1-smooth 2nd order i.d. of a rotational surface in a vicinity of its pole. Let the surface admit a field of C^1-smooth inf.deformation of 1st order with an m-th harmonic. Then for its extension $\mathbf{U}_2$ to an 2nd order inf. deformation we have the following presentations

$$\mathbf{U}_2 = a(\rho,\theta)\mathbf{k} + b(\rho,\theta)\mathbf{e} + c(\rho,\theta)\mathbf{e}'$$

where

$$\begin{aligned} a &= a_0 + a_{-2m}e^{-2im\theta} + a_{2m}e^{2im\theta}, a_{-2m} = \bar{a}_{2m} \\ b &= b_0 + b_{-2m}e^{-2im\theta} + b_{2m}e^{2im\theta}, b_{-2m} = \bar{b}_{2m} \\ c &= c_0 + c_{-2m}e^{-2im\theta} + c_{2m}e^{2im\theta}, c_{-2m} = \bar{c}_{2m}. \end{aligned}$$

The equation $d\mathbf{r}d\mathbf{U}_2 + (d\mathbf{U}_1)^2 = 0$ from (8) gives us equations for harmonics with numbers 0 and $2m$ of the field $\mathbf{U}_2$. The right sides of these equations are depending on harmonics $\alpha_m(\rho)$ and $\beta_m(\rho)$ of the 1st order inf. deformations while the left sides are exactly the same as in the equations (3)-(4) for the harmonics with the number $2m$. Supposing that we know the behavior of the harmonics with numbers m and $2m$ we can find some necessary/sufficient conditions for existence/inexistence of the 2nd order inf. deformation.

Namely we have the following theorem(given in a simple form).

Theorem. Let an analytic surface of revolution with a meridian $z = \varphi(\rho) \sim C\rho^{2k}, \rho \to 0, k \geq 2$. admit an inf. deformation with a m-th harmonic. Let the number ν_{2m} be no integer or it be an integer odd. Then for the expansibility of the m-th harmonic to a 2nd order inf. bending it is necessary and sufficient that the equation (9) has a solution $Y = m, X = m + k - 1 + 2p$ (or corresponding $\nu_m = m + 2p$) with $2p \geq k$.

A surface of revolution homeomorphic to the sphere can have a parallel orthogonal to the axe of rotation. In this case under a very large conditions to its meridian the surface will be rigid relatively to 2nd order inf. deformations. So it is interesting to study only the cases when the meridian of surface doesn't contain a point where its tangent orthogonal to the axe of rotation. With this condition we have the following

Theorem. Suppose that an analytic surface of revolution S is inf. non rigid with an analytic m-th harmonic. Let for both of poles of S the number ν_{2m} be no integer or it be an integer odd. Then in order to the field of inf. deformations of S defined by its m-th harmonic be expansible to an analytic 2nd order inf. deformation of S it is necessary and sufficient that it be expansible to a 2nd order inf. deformation in a neighebourhoods of both of poles.

Examples

1) Let's consider a surface with orders of flatness $2k_1 - 2 = 60$ (that is $k_1 = 31$) and $2k_2 - 2 = 132$ (that is $k_2 = 67$) at poles. Accordingly to the table 5 the corresponding surface can have an harmonic with the number $m = 11$. In this case near the first pole the equation (9) has a solution with $x = 91 = m + k_1 - 1 + 2p = 11 + 30 + 2p$ so $2p = 50 > k_1$ and the expansion is possible. But for the second pole we have $x = 143, 2p = 66 < k_2 = 67$ so an expansion is not possible and as the result we have that if a surface with given above orders of flatness at poles admits an inf. bending with the 11th harmonic then this inf. deformation is not expansible to a 2nd order inf. deformation.

2) Let's now $k_1 = 9, k_2 - 81$. This pair is compatible with $m = 64$. By our theorem of existence we can find a surface of revolution with these orders of flatness at poles which admits an inf. bending with a 64-th harmonic. For the corresponding equation (9) at poles we have $x = 264, 2p = 192 > k_1 = 9$ and $x = 816, 2p = 642 > k_2 = 81$ so the surface is bendable relatively of 2nd order inf. deformations.

Conclusions for 2nd order inf.bendings

1) By our method we have succeeded to find the first examples of the 1st order inf. deformations which are not expansible even locally to 2nd order inf. deformations. But the question on the existence of surfaces with local 2nd order rigidity still remains open. For this we must prove that any 1st order inf.bending of a surface is not expansible to a 2nd order infinitesimal deformations.

2) We have constructed also a first example of an analytic compact surface bendable relatively to analytical 2nd order infinitesimal deformations.