Metric Aspects of Carnot–Carathéodory Spaces and Applications

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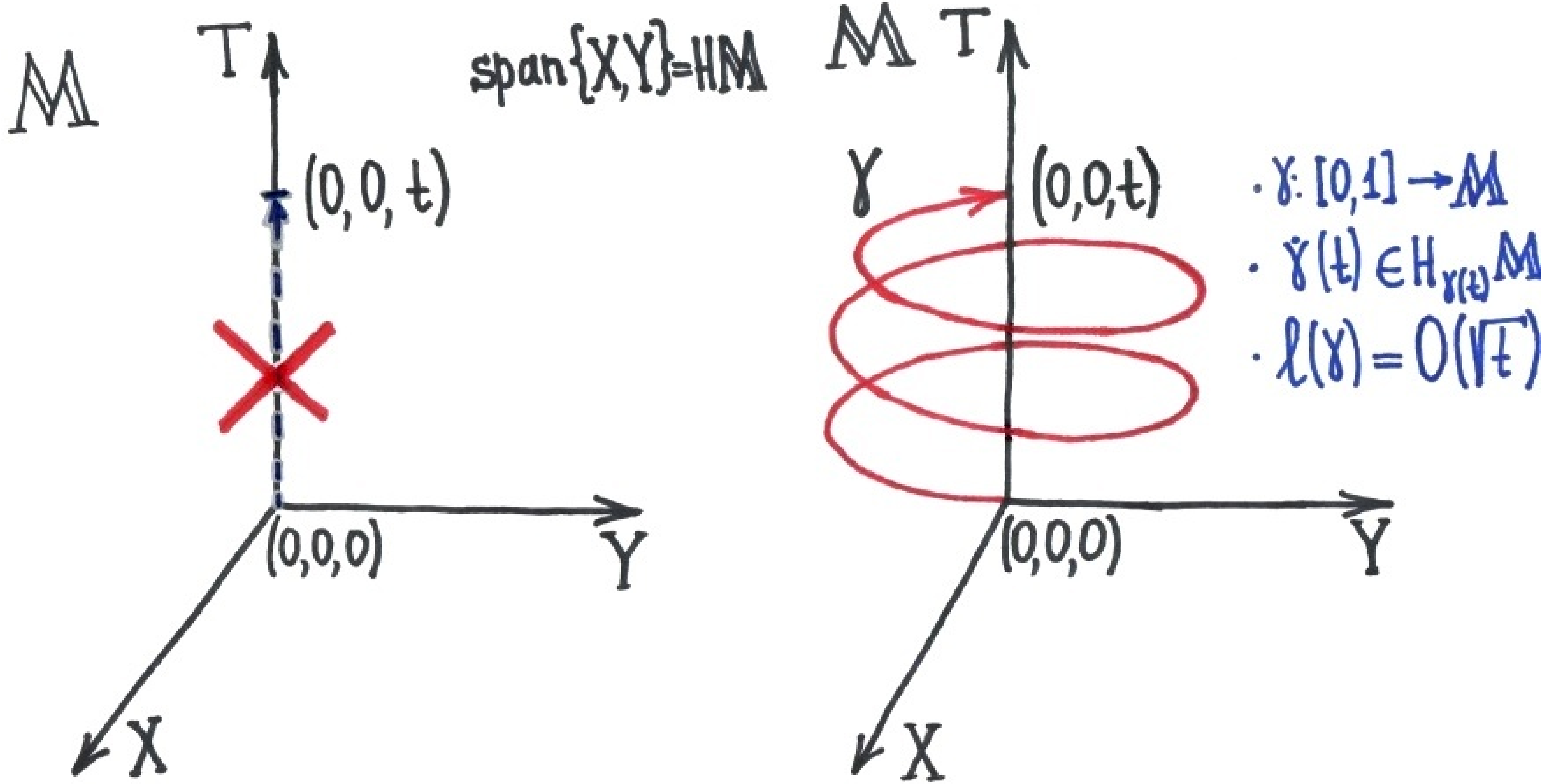


𝕄
T
(0, 0, t)
(0,0,0)
Y
X
span{X,Y}=H𝕄
𝕄
T
γ
(0,0,t)
· γ: [0,1] → 𝕄
· γ̇(t) ∈ H_{γ(t)}𝕄
· ℓ(γ) = O(√t)
(0,0,0)
Y
X

Main Results and Their Direct Corollaries

$\diamond$ new metric properties of Carnot–Carathéodory spaces

$\diamond$ extension from classical results to the minimal smoothness

$\Rightarrow$ sub-Riemannian differentiability theory (Vodopyanov)

$\Rightarrow$ non-equiregular Carnot–Carathéodory spaces (Selivanova)

$\Rightarrow$ problem on attainable set (C^1-fields) (Basalaev, Vodopyanov)

$\Rightarrow$ coarea formula for smooth contact mapping of CC-spaces

$\Rightarrow$ area formula for intrinsic Lipschitz mappings of CC-spaces

$\Rightarrow$ minimal surfaces-«graphs» on classes of Carnot groups

Carnot–Carathéodory Space (Smooth Vector Fields)

- $\mathbb{M}$ is a connected Riemannian C^∞-smooth manifold

- $\dim_{\mathrm{top}}(\mathbb{M}) = N$

- $\exists H = H\mathbb{M} \subset T\mathbb{M}$, $H\mathbb{M}$ that generates the tangent bundle $T\mathbb{M}$:

$$\begin{aligned} H = H_1 &\subsetneq \mathrm{span}\{H_1, [H_1, H_1]\} =: H_2 \\ &\subsetneq \mathrm{span}\{H_2, [H_1, H_2]\} =: H_3 \subsetneq \ldots \subsetneq H_M = T\mathbb{M} \end{aligned}$$

- $\dim H_i$'s do not depend on x (*equiregular* case; differs from Hörmander's one)

Hypoelliptic Equations

A problem: *when a distribution solution f to the equation*

$$(X_1^2 + \ldots + X_{n-1}^2 - X_n)f = \varphi \in C^\infty$$

is a smooth function?

- **Particular case: Kolmogorov's equations**

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = f$$

- physics (diffusion process), economics (arbitrage theory, some stochastic volatility models of European options), etc.

Hypoelliptic Equations

• **Hörmander (1967)**: sufficient conditions on fields $X_1, \ldots, X_n$:

Let $H_1(u) = \text{span}\{X_1(u), \ldots, X_n(u)\}$. *There exists* $M < \infty$ *s. t.*

$$\begin{aligned} H_1(u) &\subsetneq \text{span}\{H_1(u), [H_1, H_1](u)\} =: H_2(u) \\ &\subsetneq \text{span}\{H_2(u), [H_1, H_2](u)\} =: H_3(u) \subsetneq \ldots \subsetneq H_M(u) = T_u\mathbb{R}^N \end{aligned}$$

for all points $u \in D$.

• **Stein (1971)**: The program of studying of geometry of Hörmander vector fields;
description of singularities of fundamental solutions

Main Classical Results

- 1970 Goodman R. W.
- 1976 Rothschild L., Stein E.
- 1976 Metivier G.
- 1985 Nagel A., Stein E. M., Wainger S.
- 1985 Mitchell J.
- 1994 Berestovskii V.N.
- 1994 A. M. Vershik and V. Ya. Gershkovich
- 1994 M. I. Zelikin and V. F. Borisov
- 1996 Gromov M.
- 1996 Belläiche A.
- 2000 Margulis G. A., Mostow G. D.
- 2001 Agrachev A. A., Gauthier J.-P.
- 2002 Montgomery R.
- 2003 Agrachev A., Marigo A.
- 2008 J. Petitot

Applications

- subelliptic equations
- non-holonomic mechanics
- contact geometry
- physics
- robotechnics (C^1-fields are important)
- neurobiology (C^1-fields are important)

etc.

A Carnot–Carathéodory Space (C^1-Fields)

- $\mathbb{M}$ is a connected Riemannian C^∞-manifold, $\dim_{\text{top}}(\mathbb{M}) = N$
- in $T\mathbb{M}$, there is a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

- $\forall v \in \mathbb{M}\, \exists\, U(v)$ with C^1-smooth vector fields $X_1, X_2, \ldots, X_N$, s. t.
- $H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

($*$) $[H_i, H_j] \subseteq H_{i+j}$, $i, j = 1, \ldots, M$.

If $H_{j+1} = \text{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$, where $k = \left[\frac{j+1}{2}\right]$, $j = 1, \ldots, M-1$ **then** $\mathbb{M}$ is a **Carnot manifold**.

$\diamond$ $\deg X_i = \min\{k : \ X_i \in H_k\}$, $i = 1, \ldots, N$

An Example of a Carnot–Carathéodory Space with C^1-Smooth Basis Vector Fields (Answer to Gromov's Question)

- $\psi, \varphi, \xi, \eta, \omega \in C^1([a,b], \mathbb{R}) \setminus C^2$; $\psi, \varphi, \xi, \eta, \omega \neq 0$, $\frac{d\varphi}{dy} \neq 0$ on $[a,b]$

$$X = \psi(x)\partial_x + \psi(x)\varphi(y)\partial_y + \eta\Big(-\int\limits_a^y \widetilde{\xi}(x,s)\,ds + z\Big)\partial_z + \varphi(y)\omega(q)\partial_q,$$

$$Y = \partial_y + \xi\Big(-\int\limits_a^y \frac{dt}{\varphi(t)} + x\Big)\partial_z,$$

$$Z = -\psi(x)\partial_y + \omega(q)\partial_q,$$

$$T = \partial_y.$$

Here $\widetilde{\xi}(x,s) = \xi\Big(-\int\limits_a^s \frac{dt}{\varphi(t)} + x\Big)$.

$\Rightarrow$ $X \in C^1(x, y, z, q)$, $Y \in C^1(x, y)$, $Z \in C^1(x, q)$, $T \in C^\infty$

- $[X, Y] = \frac{d\varphi}{dy} \cdot Z$ and $[Y, Z] = -\frac{d\psi}{dx} \cdot T$
- $H = H_1 = \operatorname{span}\{X, Y\}$, $H_2 = \operatorname{span}\{X, Y, Z\}$, $H_3 = \operatorname{span}\{X, Y, Z, T\}$

Peculiarities and Difficulties (C^1-Smooth Vector Fields)

? What structure approximates CC-space and how?

▶ Gromov's Theorem or Baker–Campbell–Hausdorff formula...

⊖ They are unknown!

? What (quasi)metric should we use?

▶ Rashevsky–Chow Theorem (d_{cc}) / generalized $\triangle$ inequality (d_∞)...

⊖ They are unknown!

A Local Lie Group at $u \in \mathbb{M}$ (C^1-Smooth Vector Fields)

$$(\,*\,) \Rightarrow [X_i, X_j](v) = \sum_{k\colon \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v),$$

Property. Coefficients $\{c_{ijk}(u)\}_{\deg X_k = \deg X_i + \deg X_j} = \{\bar{c}_{ijk}\}$ enjoy Jacobi identity:

$$\sum_k \bar{c}_{ijk}(u)\bar{c}_{kml}(u) + \sum_k \bar{c}_{mik}(u)\bar{c}_{kjl}(u) + \sum_k \bar{c}_{jmk}(u)\bar{c}_{kil}(u) = 0$$

for all $i, j, m, l = 1, \ldots, N$, and $\bar{c}_{ijk} = -\bar{c}_{jik}$ for all $i, j, k = 1, \ldots, N$
$\Rightarrow$ <u>nilpotent Lie algebra</u>

basis vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ *in* $\mathbb{R}^N$ *are chosen in a such a way that* $\exp = \mathrm{Id}$. This implies $(\widehat{X}_i^u)'(0) = e_i$, $i = 1, \ldots, N$

$\Rightarrow$ corresponding Lie group: <u>nilpotent Lie group</u>

Local Homogeneous Lie Group $\mathcal{G}^u\mathbb{M}$

$\mathcal{G} \subset \mathbb{M}$: a neighborhood of u with basis fields $\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$

$\theta_u : U(0) \to \mathcal{G}$ group isomorphism $\Rightarrow$ structure of local Lie group

For $x = \exp\Big(\sum_{i=1}^N x_i \widehat{X}_i^u\Big)(u)$, $y = \exp\Big(\sum_{i=1}^N y_i \widehat{X}_i^u\Big)(u)$, define

$$x \cdot y = \exp\Big(\sum_{i=1}^N z_i \widehat{X}_i^u\Big)(u),$$

where

$$z_i = x_i + y_i + \sum_{|\mu+\beta|_h = \deg X_i,\ \mu,\beta>0} F^i_{\mu,\beta}(u) x^\mu y^\beta.$$

- fields $\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$ are left-invariant

Distance Functions

- If $w = \exp\Big(\sum_{i=1}^{N} w_i \widehat{X}_i^u\Big)(v)$ then $d_\infty^u(v,w) = \max_{i=1,\dots,N}\{|w_i|^{\frac{1}{\deg X_i}}\}$
- d_∞^u is a quasimetric: $\exists C = C(\mathcal{U})$ such that

$$d_\infty^u(v,w) \le C(d_\infty^u(v,z) + d_\infty^u(z,w)) \quad \forall w,v,z \in \mathcal{U}$$

- If $v = \exp\Big(\sum_{i=1}^{N} v_i \widehat{X}_i^u\Big)(u)$ then $\Delta_\varepsilon^u(v) = \exp\Big(\sum_{i=1}^{N} \varepsilon^{\deg X_i} v_i \widehat{X}_i^u\Big)(u)$
- If $w = \exp\Big(\sum_{i=1}^{N} w_i X_i\Big)(v)$ then $d_\infty^u(v,w) = \max_{i=1,\dots,N}\{|w_i|^{\frac{1}{\deg X_i}}\}$
- Main Result $\Rightarrow$ d_∞ is a quasimetric

Main Result

- $\mathbb{M}$: a CC-space with $C^{1,\alpha}$-smooth basis vector fields
- $\alpha \in [0,1]$; $\alpha = 0 \Rightarrow$ the vector fields are just C^1-smooth

Theorem. $\forall\ w \in \mathbb{M}\ \exists\ \mathcal{O} \ni w$, $\mathcal{O} \Subset \mathbb{M}$, such that $\forall\ u, x \in \mathcal{O}$ and $\forall$ $q = 1, \ldots, N$ the representation holds

$$X_q(\Delta_\varepsilon^u x) = \sum_{p=1}^{N} a_{p,q}^u(\Delta_\varepsilon^u x)\widehat{X}_p^u(\Delta_\varepsilon^u x),$$

where

$$a_{p,q}^u(\Delta_\varepsilon^u x) = \begin{cases} O(\varepsilon), & \deg X_p < \deg X_q, \\ \delta_{pq} + O(\varepsilon), & \deg X_p = \deg X_q, \\ O(\varepsilon^{\alpha + \deg X_p - \deg X_q}), & \deg X_p > \deg X_q,\ \ \alpha > 0, \\ o(\varepsilon^{\deg X_p - \deg X_q}), & \deg X_p > \deg X_q,\ \ \alpha = 0. \end{cases}$$

Comparison of Local Geometries (Local Groups)

- $\mathcal{U} \Subset \mathbb{M}$ is an arbitrary neighborhood described in the main result
- $d_\infty(w,v)$ and $d_\infty^u(w,v)$ are well-defined $\forall u,v,w \in \mathcal{U}$

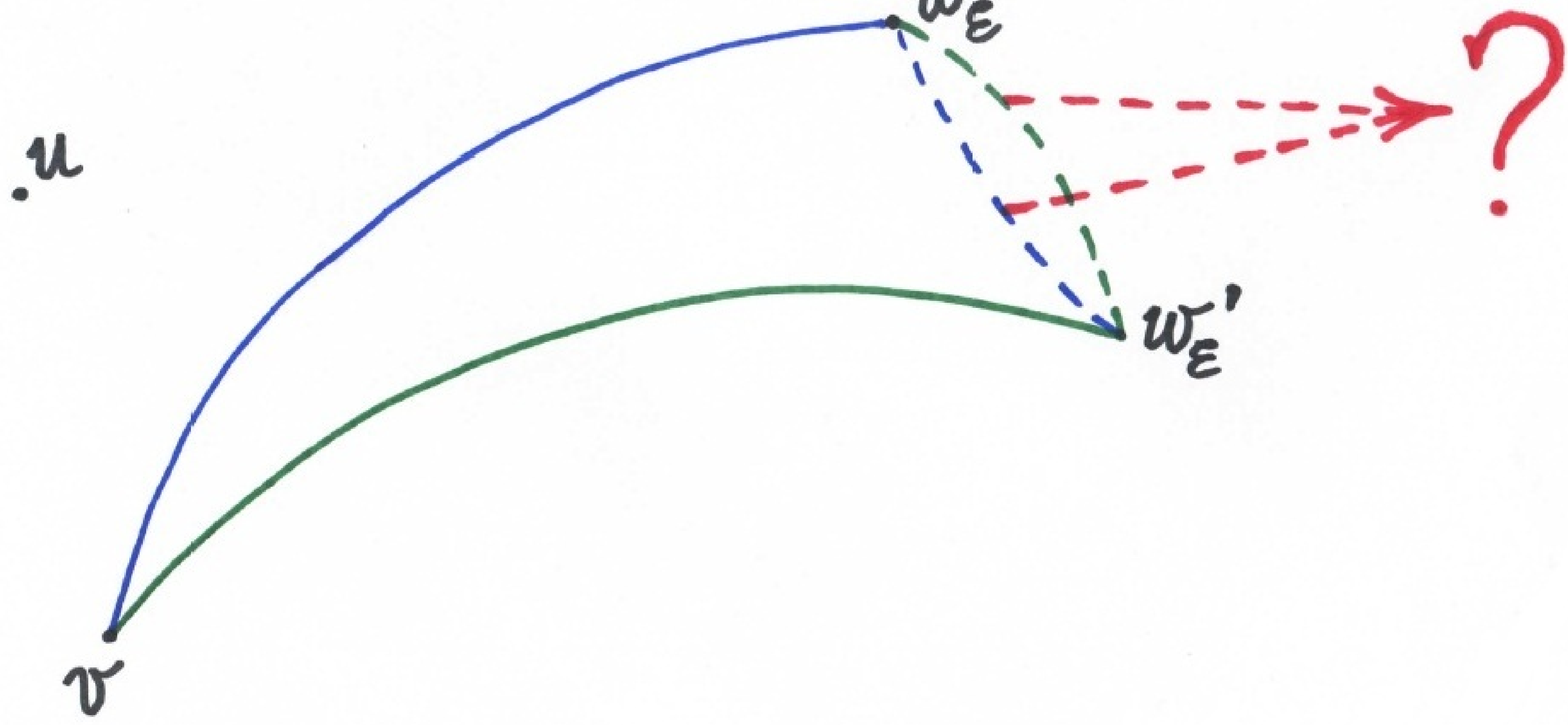
Theorem. $\forall u,v,u' \in \mathcal{U}$, $w'_\varepsilon = \gamma(1)$, $w_\varepsilon = \widehat{\gamma}(1)$, where $\gamma, \widehat{\gamma} : [0,1] \to \mathbb{M}$, $\gamma, \widehat{\gamma} \subset \mathrm{Box}(u,\varepsilon)$, $u' \in \mathrm{Box}(u,\varepsilon)$ such that $\gamma(0) = \widehat{\gamma}(0) = v$ and

$$\dot{\gamma}(t) = \sum_{i=1}^{N} b_i(t) \widehat{X}_i^{u'}(\gamma(t)),\ \dot{\widehat{\gamma}}(t) = \sum_{i=1}^{N} b_i(t) \widehat{X}_i^{u}(\gamma(t)),\ \int_0^1 |b_i(t)|\, dt < S\varepsilon^{\deg X_i},$$

$S < \infty$, $i = 1, \ldots, N$, we have

$$\max\{d_\infty^{u'}(w_\varepsilon, w'_\varepsilon), d_\infty^{u}(w_\varepsilon, w'_\varepsilon)\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

with $O(1)$ and $o(1)$ to be uniform as $\varepsilon \to 0$.



u
v
w_ε
w_ε'
?

Application of the Estimate for Groups

- Let $b_i(t) = w_i$, $i = 1, \ldots, N$

$\Rightarrow$ $w_\varepsilon = \exp\Big(\sum_{i=1}^{N} w_i \widehat{X}_i^u\Big)(v)$, $w'_\varepsilon = \exp\Big(\sum_{i=1}^{N} w_i \widehat{X}_i^{u'}\Big)(v)$

($\triangle'$) $d_\infty^p(w,v) \le d_\infty^p(w,u) + C_{\mathcal{U}} d_\infty^p(u,v)$ is true on $\mathcal{G}^p\mathbb{M}$, $p \in \mathcal{U}$

$$\Rightarrow \ |d_\infty^u(v,w_\varepsilon) - d_\infty^{u'}(v,w_\varepsilon)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

If $u' = v \Rightarrow |d_\infty^u(v,w_\varepsilon) - d_\infty(v,w_\varepsilon)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$

Thus, <u>we can compare d_∞^u and d_∞</u> for points from $\mathcal{U} \subset \mathbb{M}$

Comparison of Local Geometries

- $\mathcal{U} \Subset \mathbb{M}$ is an arbitrary neighborhood small enough
- $d_\infty(w,v)$ and $d_\infty^u(w,v)$ are well-defined $\forall u,v,w \in \mathcal{U}$

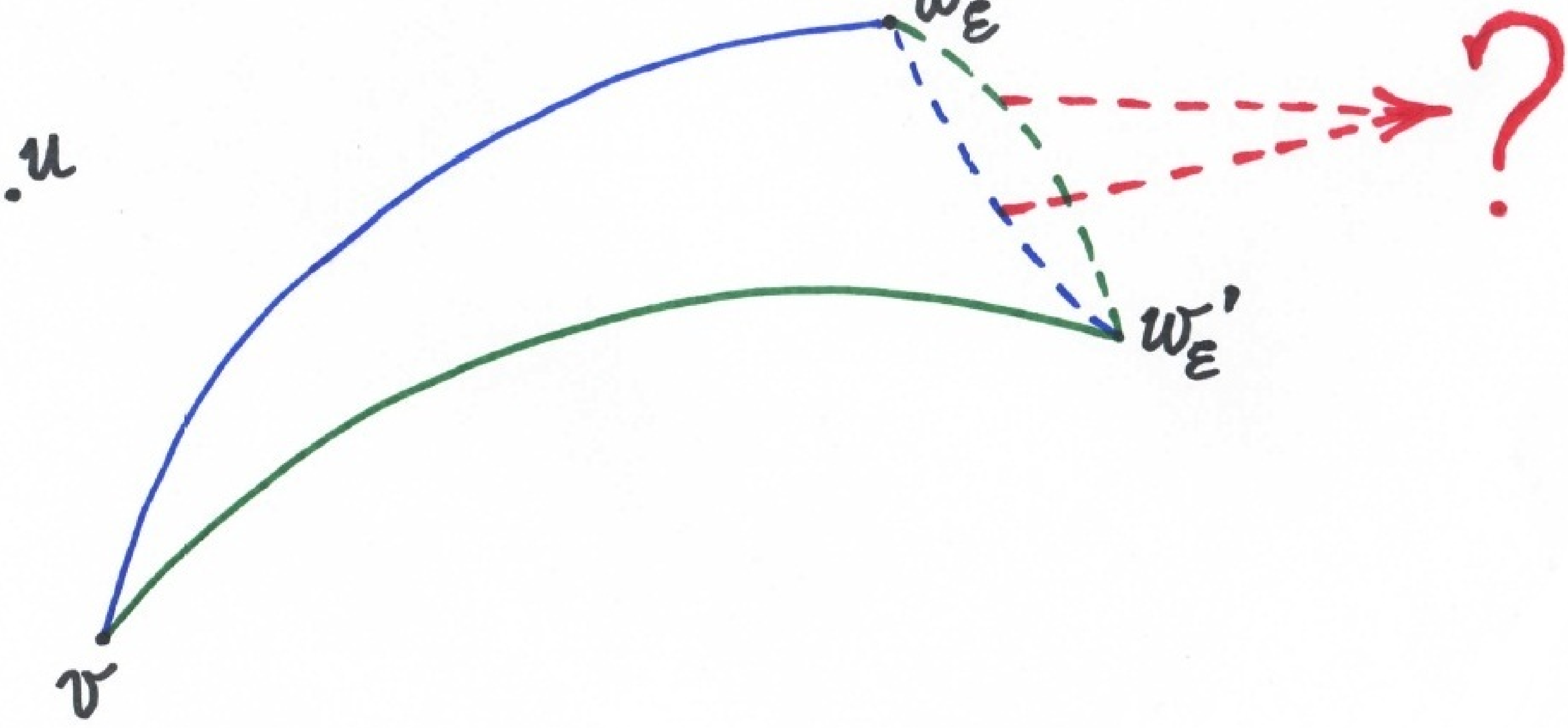
Theorem. $\forall u,v \in \mathcal{U}$, $w_\varepsilon = \gamma(1)$, $w'_\varepsilon = \widehat{\gamma}(1)$, where $\gamma, \widehat{\gamma} : [0,1] \to \mathbb{M}$, $\gamma, \widehat{\gamma} \subset \mathrm{Box}(u,\varepsilon)$ such that $\gamma(0) = \widehat{\gamma}(0) = v$ and

$$\dot{\gamma}(t) = \sum_{i=1}^{N} b_i(t) X_i(\gamma(t)),\ \dot{\widehat{\gamma}}(t) = \sum_{i=1}^{N} b_i(t) \widehat{X}_i^u(\gamma(t)),\ \int_0^1 |b_i(t)|\, dt < S\varepsilon^{\deg X_i},$$

$S < \infty$, $i = 1, \ldots, N$, we have

$$\max\{d_\infty(w_\varepsilon, w'_\varepsilon), d_\infty^u(w_\varepsilon, w'_\varepsilon)\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

with $O(1)$ and $o(1)$ to be uniform as $\varepsilon \to 0$.



u
v
w_ε
w_ε'
?

Estimates for Chains (I)

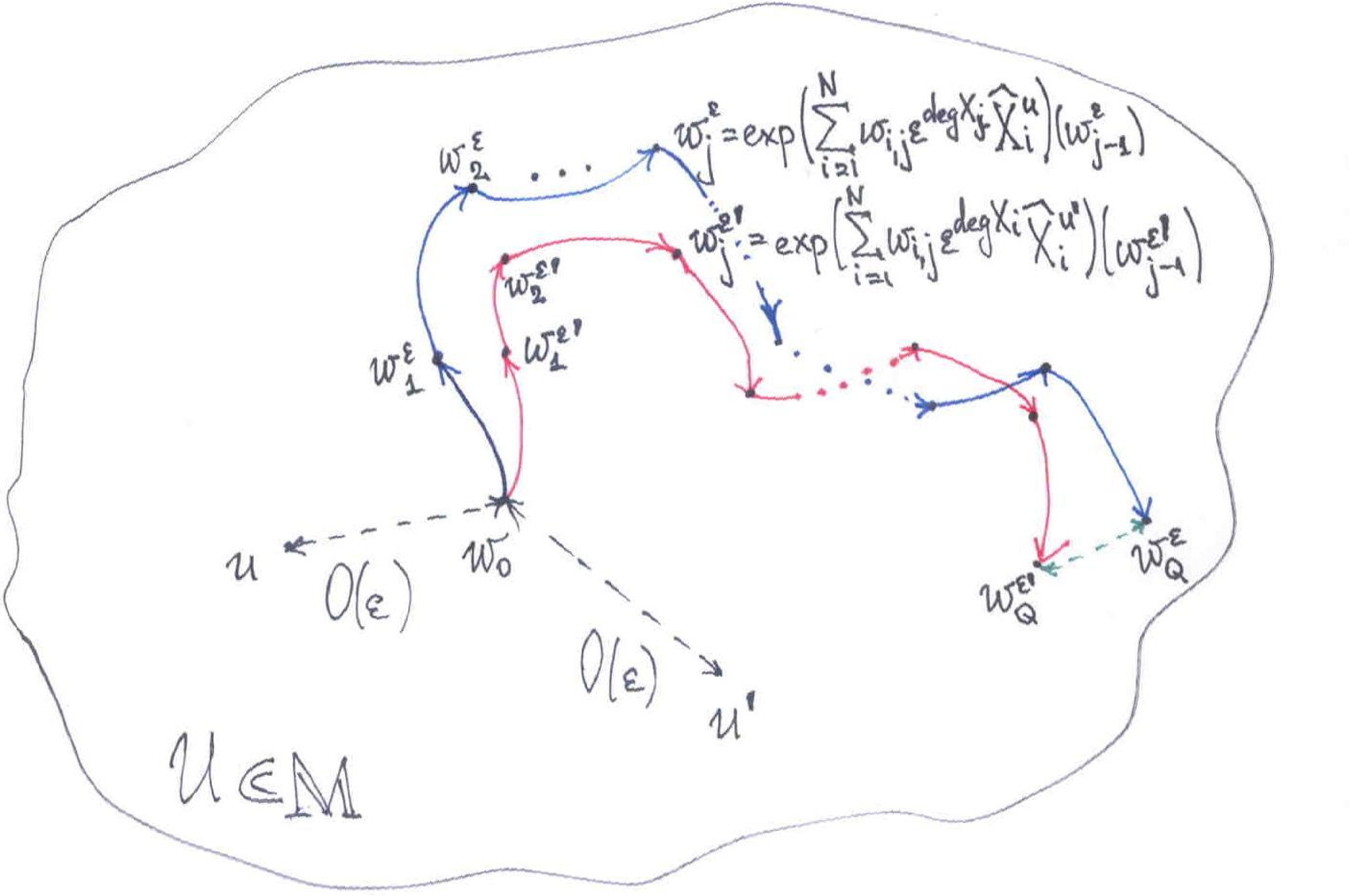
Theorem. For $Q \in \mathbb{N}$, consider $u \in \mathcal{U}$, $u', w_0 \in \mathsf{Box}(u, \varepsilon)$ and

$$w_j^\varepsilon = \exp\left(\sum_{i=1}^{N} w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(w_{j-1}^\varepsilon),\ w_j^{\varepsilon\prime} = \exp\left(\sum_{i=1}^{N} w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\right)(w_{j-1}^{\varepsilon\prime})$$

$w_0^{\varepsilon\prime} = w_0^\varepsilon = w_0$, $j = 1, \dots, Q$. (Here $Q \in \mathbb{N}$ is such that all these points and lines lie in $\mathcal{U} \subset \mathbb{M}$ for sufficiently small $\varepsilon > 0$.) Then

$$\max\{d_\infty^u(w_Q^\varepsilon, w_Q^{\varepsilon\prime}), d_\infty^{u'}(w_Q^\varepsilon, w_Q^{\varepsilon\prime})\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0. \end{cases}$$

Here $O(1)$ is uniform in $u \in \mathcal{U}$, $u', w_0 \in \mathsf{Box}(u, \varepsilon)$ and $\{w_{i,j}\}$, $i = 1, \dots, N$, $j = 1, \dots, Q$, from some compact neighborhood of 0, and depends on Q and $\{F_{\mu,\beta}^j|\mathcal{U}\}_{j,\mu,\beta}$.



$w_j^{\varepsilon} = \exp\left(\sum_{i=1}^{N} w_{i,j}\varepsilon^{\deg X_i}\widehat{X}_i^{u}\right)(w_{j-1}^{\varepsilon})$
$w_j^{\varepsilon'} = \exp\left(\sum_{i=1}^{N} w_{i,j}\varepsilon^{\deg X_i}\widehat{X}_i^{u'}\right)(w_{j-1}^{\varepsilon'})$
w_2^{ε}
w_1^{ε}
$w_2^{\varepsilon'}$
$w_1^{\varepsilon'}$
w_0
u
$O(\varepsilon)$
$O(\varepsilon)$
u'
w_Q^{ε}
$w_Q^{\varepsilon'}$
$U \Subset M$

Estimates for Chains (II)

Theorem. For $Q \in \mathbb{N}$, consider $u \in \mathcal{U}$, $u', w_0 \in \mathrm{Box}(u, \varepsilon)$ and

$$w_j^\varepsilon = \exp\left(\sum_{i=1}^N w_{i,j}\varepsilon^{\deg X_i}\widehat{X}_i^u\right)(w_{j-1}^\varepsilon),\ w_j^{\varepsilon\prime} = \exp\left(\sum_{i=1}^N w_{i,j}\varepsilon^{\deg X_i}X_i\right)(w_{j-1}^{\varepsilon\prime}),$$

$w_0^{\varepsilon\prime} = w_0^\varepsilon = w_0$, $j = 1, \ldots, Q$. (Here $Q \in \mathbb{N}$ is such that all these points and lines lie in $\mathcal{U} \subset \mathbb{M}$ for sufficiently small $\varepsilon > 0$.) Then

$$\max\{d_\infty^u(w_Q^\varepsilon, w_Q^{\varepsilon\prime}), d_\infty(w_Q^\varepsilon, w_Q^{\varepsilon\prime})\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0. \end{cases}$$

Here $O(1)$ is uniform in $u \in \mathcal{U}$, $w_0 \in \mathrm{Box}(u, \varepsilon)$ and $\{w_{i,j}\}$, $i = 1, \ldots, N$, $j = 1, \ldots, Q$, from some compact neighborhood of 0, and depends on Q and $\{F_{\mu,\beta}^j|\mathcal{U}\}_{j,\mu,\beta}$.

Application: Rashevsky–Chow Theorem

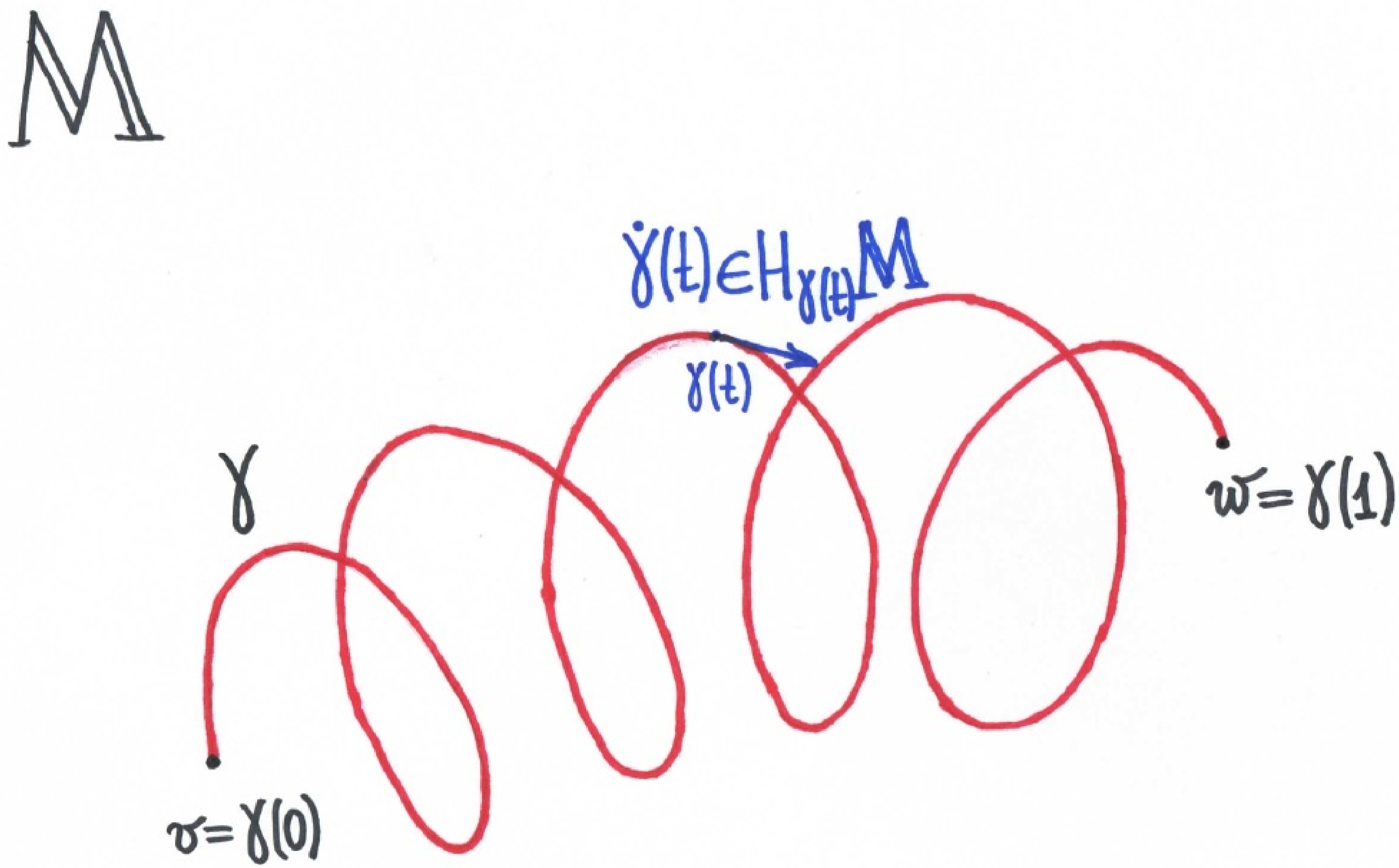
Suppose that $\mathbb{M}$ is a Carnot manifold

Theorem. *Any two points* $x, y \in \mathbb{M}$ *can be connected by a horizontal curve* γ *(i. e.,* $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ *for almost all* $t \in [0, 1]$*).*

- Vodopyanov, Basalaev: C^1-smooth case

Definition. Carnot–Carathéodory distance d_{cc} between points x and y equals

$$\inf\{\ell(\gamma) : \gamma(0) = x,\, \gamma(1) = y,\, \dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}\}$$


$\mathbb{M}$
$\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$
$\gamma(t)$
γ
$w = \gamma(1)$
$v = \gamma(0)$

Corollaries

- d_∞ is a quasimetric
- d_∞ and d_{cc} are locally bi-Lipschitz equivalent (Ball-Box Theorem):

$$c(\mathcal{U})d_\infty(x,y) \le d_{cc}(x,y) \le C(\mathcal{U})d_\infty(x,y),$$

- $\dim_{\mathcal{H}} \mathbb{M} = \dim H_1 + \sum\limits_{i=2}^{M} i(\dim H_i - \dim H_{i-1}) := \nu$
- Local Approximation Theorems: $u', v, w \in B_{cc}(u, \varepsilon) \Rightarrow$

$$|d_\infty(v,w) - d_\infty^u(v,w)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0; \end{cases}$$

$$|d_{cc}(v,w) - d_{cc}^u(v,w)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0. \end{cases}$$

  - sub-Riemannian analog of manifold's approximation by a tangent space

Weighted Carnot–Carathéodory Spaces

- Weights instead of degrees: $l_1 < \ldots < l_Q$, $l_k \in \mathbb{N}$, $k = 1, \ldots, Q$
- «usual» Carnot–Carathéodory space: $l_k = k$, $Q = M$

- $\{X_i\}_{i=1}^N \in C^M$: S. Selivanova, definition and properties (+non-equiregularity)

- $[H_i, H_j] \subseteq H_m$, $m = \max\{p : \ l_i + l_j \geq l_p\}$, $i, j = 1, \ldots, Q \Leftrightarrow$

$$[X_i, X_j](v) = \sum_{k:\,\mathrm{wgt}\,X_k \leq \mathrm{wgt}\,X_i + \mathrm{wgt}\,X_j} c_{ijk}(v) X_k(v) \quad (\{X_i\}_{i=1}^N \in C^1)$$

- $\{c_{ijk}(u)\}_{\mathrm{wgt}\,X_k = \mathrm{wgt}\,X_i + \mathrm{wgt}\,X_j} = \{\bar{c}_{ijk}\} \Rightarrow$ local Lie group

- comparison estimates: $\begin{cases} O(\varepsilon^{1+\frac{\min\{\alpha l_1, 1\}}{l_Q}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$

Main Applications

• *Sub-Riemannian* *Differentiability Theory*: Rademacher- and Stepanov-type theorems on sub-Riemannian differentiability of mappings of Carnot manifolds (S. Vodopyanov)

• *Geometric Measure Theory on metric structures*: area formula for intrinsically Lipschitz mappings, coarea formula for C^{M+1}-smooth mappings of Carnot manifolds, criteria for minimal surfaces (M. Karmanova; S. Vodopyanov and и M. Karmanova)

• *Geometry of non-equiregular (and weighted) Carnot–Carathéodory spaces* (S. Selivanova)

• *Rashevsky–Chow Theorem for C^1-smooth case* $\Rightarrow$ problem on attainable sets (S. Basalaev, S. Vodopyanov)

Sub-Riemannian Differentiability

Definition [S. Vodopyanov]. Let $\mathbb{M}$ and $\widetilde{\mathbb{M}}$ be CC-spaces, and let $D \subset \mathbb{M}$. A mapping $\varphi : (D, d_\infty) \to (\widetilde{\mathbb{M}}, \tilde{d}_\infty)$ is *hc-differentiable* at $u \in \mathbb{M}$, if there exists a horizontal homomorphism

$$L_u : (\mathcal{G}^u, d_\infty^u) \to (\mathcal{G}^{\varphi(u)}, d_\infty^{\varphi(u)})$$

of local Carnot groups such that

$$\tilde{d}_\infty(\varphi(w), L_u(w)) = o(d_\infty(u, w)),\ \ D \cap \mathcal{G}^u \ni w \to u.$$

- $\varphi : D \to \widetilde{\mathbb{G}}$, $D \subset \mathbb{G}$ $\Rightarrow$ $\mathcal{P}$-differentiability [Pansu]
- hc-differential of φ at u is denoted by $\widehat{D}\varphi(u)$

Theorem (Vodopyanov). *Let $\mathbb{M}$ be a Carnot manifold, $D \subset \mathbb{M}$ be a measurable set, and $\widetilde{\mathbb{M}}$ be a Carnot–Carathéodory space. Every intrinsically Lipschitz mapping $\varphi : D \to \widetilde{\mathbb{M}}$ is hc-differentiable almost everywhere.*

Sub-Riemannian Area Formula

Classical result: $\varphi : D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$, $n \le m$

$$\int_D f(y)\sqrt{\mathrm{det}\Big(D\varphi(y)^* D\varphi(y)\Big)}\, d\mathcal{H}^n(y) = \int_{\mathbb{R}^m} \sum_{y:y\in\varphi^{-1}(x)} f(y)\, d\mathcal{H}^n(x)$$

Sub-Riemannian Result: $\mathbb{M}$ is a Carnot manifold, $\widetilde{\mathbb{M}}$ is a CC-space, $D \subset \mathbb{M}$, $\varphi \in \mathrm{Lip}^{SR}(D, \widetilde{\mathbb{M}})$, $\dim H_1 \le \dim \widetilde{H}_1$;

$$\int_{\mathbb{M}} f(y)\sqrt{\mathrm{det}(\widehat{D}\varphi(x)^* \widehat{D}\varphi(x))}\, d\mathcal{H}^\nu(y) = \int_{\widetilde{\mathbb{M}}} \sum_{y:\, y\in\varphi^{-1}(x)} f(y)\, d\mathcal{H}^\nu(x)$$

Sub-Riemannian Coarea Formula

Classical result: $\varphi : \mathbb{R}^n \to \mathbb{R}^k$, $n \geq k$

$$\int_{\mathbb{R}^n} f(x)\sqrt{\det\Big(D\varphi(x)D\varphi(x)^*\Big)}\,d\mathcal{H}^n(x) = \int_{\mathbb{R}^k} d\mathcal{H}^k(z) \int_{\varphi^{-1}(x)} f(u)\,d\mathcal{H}^{n-k}(u)$$

Sub-Riemannian Result: $\mathbb{M}$ is a Carnot manifold, $\widetilde{\mathbb{M}}$ is a CC-space, $\varphi \in C^{M+1}(\mathbb{M}, \widetilde{\mathbb{M}})$ is a contact mapping, $\dim H_1 \geq \dim \widetilde{H}_1$, $\dim H_i - \dim H_{i-1} \geq \dim \widetilde{H}_i - \dim \widetilde{H}_{i-1}$, $i = 2, \ldots, M$;

$$\int_{\mathbb{M}} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) f(x)\,d\mathcal{H}^{\nu}(x) = \int_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\widetilde{\nu}}(z) \int_{\varphi^{-1}(z)} f(u)\,d\mathcal{H}^{\nu-\widetilde{\nu}}(u),$$

where $\mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) = \sqrt{\det(\widehat{D}\varphi(x)\widehat{D}\varphi(x)^*)} \cdot C_\omega$

C_ω depends on factors in the definitions in Hausdorff measure

Main Publications

1. *Karmanova M. B.* Fine properties of basis vector fields on Carnot–Carathéodory spaces under minimal assumptions on smoothness // Sib. Math. Zh., 2014. V. 55, № 1. P. 87–99.

2. *Karmanova M. B.* Area formula for Lipschitz mappings of Carnot–Carathéodory spaces // Izvestiya RAS, 2014 (accepted)

3. *Karmanova M., Vodopyanov S.* On Local Approximation Theorem on Equiregular Carnot-Caratheodory Spaces // Proceedings of INDAM "Meeting on Geometric Control and Sub-Riemannian Geometry", Cortona, May 2012. Springer INdAM Series, 2014 (accepted)

4. *Karmanova M., Vodopyanov S.* A Coarea Formula for Smooth Contact Mappings of Carnot-Caratheodory Spaces // Acta Applicandae Mathematicae, 2013. V. 128, № 1. P. 67–111.

5. *Karmanova M.B.* Graphs of Lipschitz functions and minimal surfaces on Carnot groups // Sib. Math. Zh., 2012. V. 53, № 4. P. 839–861.

6. *Karmanova M., Vodopyanov S.* An Area Formula for Contact C^1-Mappings of Carnot Manifolds // Complex Variables and Elliptic Equations, 2010. V. 55, Issue I-III. P. 317–329.

THANK YOU!