Combinatorial model of the Lipschitz metric for punctured surfaces

Vladimir Shastin

Moscow State University

September 2014
Geometry days

Outline

1. Algorithmic problems for mapping class groups

Outline

1. Algorithmic problems for mapping class groups
2. Complexity functions on mapping class groups

Outline

1. Algorithmic problems for mapping class groups
2. Complexity functions on mapping class groups
3. Connection with metrics on Teichmüller spaces

Mapping class group

Definitions and examples

$S^n_{g,k}$ — a compact orientable surface of genus g with k boundary components and n punctures.

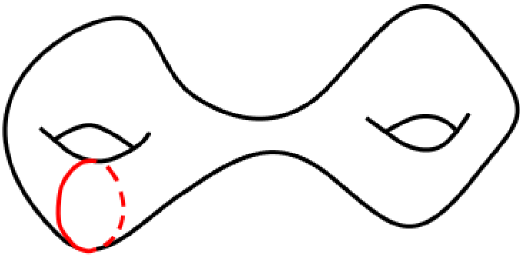
$\mathrm{MCG}(S^n_{g,k}) = \mathsf{Diff}^+(S^n_{g,k}, \partial S^n_{g,k})/\mathsf{Diff}_0(S^n_{g,k}, \partial S^n_{g,k})$.

Some examples of mapping class groups:

1. $\mathrm{MCG}(S^0_{0,0}) = \mathrm{MCG}(S^0_{0,1}) = 1$.
2. $\mathrm{MCG}(S^0_{0,2}) = \mathbb{Z}$.
3. $\mathrm{MCG}(S^0_{1,0}) = \mathrm{MCG}(S^1_{1,0}) = \mathsf{SL}_2(\mathbb{Z})$.
4. $\mathrm{MCG}(S^n_{0,1}) = B_n$.

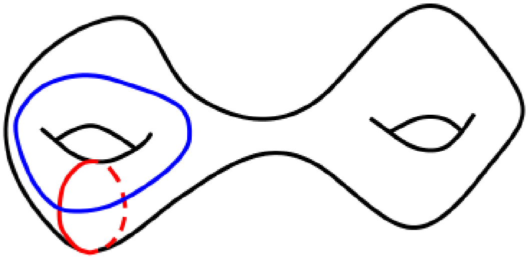
Dehn twist

To a simple essential curve α on S corresponds the non-trivial element $t_\alpha \in \mathrm{MCG}(S)$ called Dehn twist.



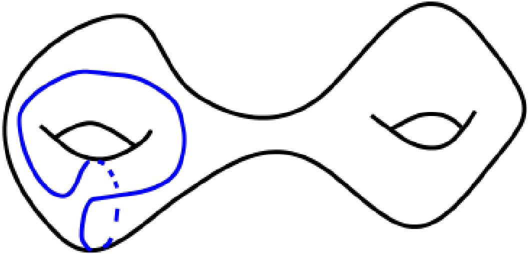
Dehn twist

To a simple essential curve α on S corresponds the non-trivial element $t_\alpha \in \mathrm{MCG}(S)$ called Dehn twist.



Dehn twist

To a simple essential curve α on S corresponds the non-trivial element $t_\alpha \in \mathrm{MCG}(S)$ called Dehn twist.



Dehn-Lickorish theorem

Theorem (Dehn,Lickorish)

If S is closed, then $\mathrm{MCG}(S)$ is generated by a finite number of Dehn twists.

Definition

An element $h \in \mathrm{MCG}(S)$ is called fractional power of a Dehn twist if there are $n \in \mathbb{Z}$, $m \in \mathbb{Z} \setminus 0$ and a Dehn twist t such that $h^m = t^n$.

Corollary

$\mathrm{MCG}(S)$ is generated by a finite number of fractional Dehn twists.

Algorithmic problems for $\mathrm{MCG}(S^n_{g,k})$

Theorem (Grossman)

The word problem in $\mathrm{MCG}(S^n_{g,k})$ is solvable.

Theorem (Hemion)

The conjugacy problem in $\mathrm{MCG}(S^n_{g,k})$ is solvable.

Remark

For braid groups the word problem was solved by Artin and Markov and conjugacy problems — by Garside and Makanin.

Fast and not so fast algorithms

Theorem (Mosher)

There exists a quadratic-time algorithm to solve the word problem in $\mathrm{MCG}(S^n_{g,k})$.

Remark

For braid groups a quadratic-time algorithm to solve the word problem was first constructed by Thurston.

Theorem (Masur-Minsky, Jing Tao)

There is a exponential-time algorithm to solve the conjugacy problem in $\mathrm{MCG}(S^n_{g,k})$.

Word length function

How to measure a complexity of an element in a finitely generated group? The standard way is to use the word length function on a group.

Definition

If $\mathcal{A}$ is a finite set of generators in a group G, then the word length $\mathrm{wl}_{\mathcal{A}}(g)$ of an element $g \in G$ with respect to $\mathcal{A}$ is defined as follows:

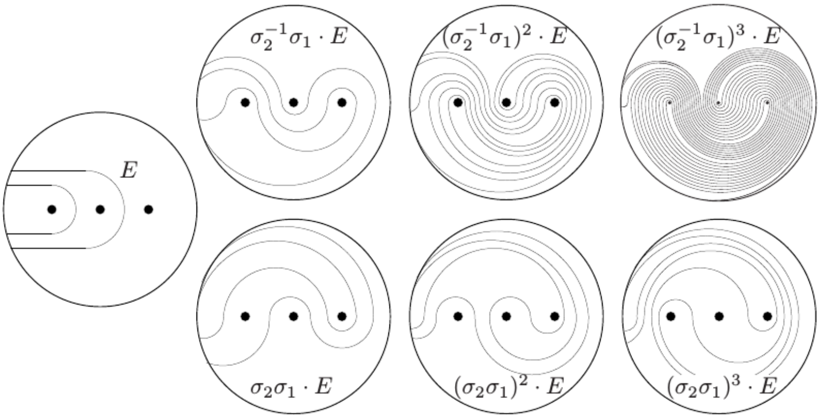
$$\mathrm{wl}_{\mathcal{A}}(g) = \min_{\substack{g=a_1^{k_1}\ldots a_m^{k_m} \\ a_1,\ldots,a_m \in \mathcal{A} \\ k_1,\ldots,k_m \in \mathbb{Z}}} \sum_{i=1}^{m} |k_i|.$$

In all previously mentioned theorems complexity of the input is assumed to be its word length.

But is the word length function is an adequate measure of complexity for mapping classes?

Elements in B_3 of equal word length

The picture from the work of Dynnikov and Wiest



Elements in $\mathrm{SL}(2,\mathbb{Z})$ of equal word length

Generators: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Words of length 100: $W_1 = A^{100}$, $W_2 = (AB)^{50}$

Corresponding matrices:

$$M_1 = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 573147844013817084101 & 354224848179261915075 \\ 354224848179261915075 & 218922995834555169026 \end{pmatrix}$$

Geometric complexity

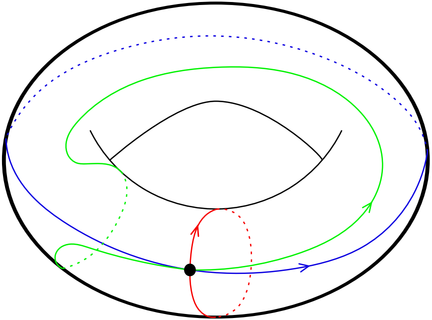
Another way to define a complexity function on the mapping class groups is to measure the entanglement of some simple curves or arcs under the action of diffeomorphisms.
For a surface S with non-empty set of punctures we can consider its ideal triangulation T and measure the complexity of a mapping class φ as follows:

$$c_T(\varphi) = \log_2 \langle \varphi(T), T \rangle,$$

where $\langle \varphi(T), T \rangle$ is the geometric intersection number of the triangulations $\varphi(T)$ and T.

Geometric complexity of Dehn twist

If T is a triangulation of a punctured torus shown below and φ is n-th power of the Dehn twist around red curve, then $c_T(\varphi) = \log_2(4n-3)$.



Geometric complexity in $\mathrm{SL}(2,\mathbb{Z})$

In fact if we consider $\mathrm{SL}(2,\mathbb{Z})$ as a mapping class group of punctured torus, then the geometric complexity defined for any triangulation is comparable with the following complexity function on matrices:

$$c_M\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \log_2(|a|+1)(|b|+1)(|c|+1)(|d|+1).$$

Here we say that two nonnegative functions f_1, f_2 on a group G are comparable if there exist $K > 1$ and $C > 0$ such that for every $g \in G$

$$\frac{1}{K} f_1(g) - C < f_2(g) < K f_1(g) + C.$$

Zipped word length

Definition

Let G be a finitely generated group and $\mathcal{A}$ be some finite generating set of G. Then the zipped word length of $g \in G$ is defined as follows:

$$\operatorname{zwl}_{\mathcal{A}}(g) = \min_{\substack{a_1^{k_1}\dots a_m^{k_m}=g \\ a_1,\dots,a_m\in\mathcal{A} \\ k_1,\dots,k_m\in\mathbb{Z}}} \sum_{i=1}^{m} \log_2(|k_i|+1).$$

We remark that unlike word length functions zipped word length functions corresponding to different generating sets may be not comparable.

Zipped word length for $\mathrm{SL}(2,\mathbb{Z})$

For generating set $\mathcal{A} = \left\{ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \right\}$ of $\mathrm{SL}(2,\mathbb{Z})$ the zipped word length function $\mathrm{zwl}_{\mathcal{A}}$ is comparable with the matrix complexity function c_M. It's not hard to show using fast Euclid's algorithm for $\mathrm{SL}(2,\mathbb{Z})$.
But if we take $\mathcal{A}' = \left\{ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \right\}$ as generating set, then $\mathrm{zwl}_{\mathcal{A}}$ is not comparable with c_M. This is so, because entries of n-th power of the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ grow exponentially as $n \to \infty$.

Admissible generating sets

For general mapping class groups generating sets like the set $\mathcal{A}$ from the previous slide are called admissible:

Definition

A finite generating set $\mathcal{A}$ of $\mathrm{MCG}(S)$ is called admissible if it has the following properties:

1. every element in $\mathcal{A}$ is a fractional power of a Dehn twist;
2. every Dehn twist is conjugate to a fractional power of an element from $\mathcal{A}$.

An algebraic analogue of the geometric complexity

Theorem (Dynnikov)

For any ideal triangulation T of S and an admissible generating set $\mathcal{A}$ of $\mathrm{MCG}(S)$ the geometric complexity function c_T and the zipped word length function $\mathrm{zwl}_{\mathcal{A}}$ are comparable.

So with respect of the zipped word length of some mapping classes become simpler. But can we effectively solve the word problem with respect to this complexity function?

Solution of the word problem

Yes.

Theorem (Dynnikov)

Let S be a surface with non-empty set of punctures and $\mathcal{A}$ be an admissible generating set for $\mathrm{MCG}(S)$. Then the word problem in $\mathrm{MCG}(S)$ is solvable in polynomial time with respect to $zwl_{\mathcal{A}}$.

Remark

For braid groups the similar result was earlier obtained by Dynnikov and Wiest.

Mapping class groups and Teichmüller spaces

From the zipped word length function $\mathrm{zwl}_{\mathcal{A}}$ we can naturally obtain the right-invariant metric $\rho_{\mathcal{A}}$ on the mapping class group of the surface. It turns out that this metric is closely related to metrics on the mapping class group coming from its action on the Teichmüller space of the surface.

Definition

The Teichmüller space of $S^n_{g,0}$ denoted by $\mathcal{T}(S^n_{g,0})$ is the set of isotopy classes of hyperbolic structures on the surface.

$\mathrm{MCG}(S^n_{g,0})$ naturally acts (by pullback of hyperbolic structure) on the Teichmüller space. This action is properly discontinious.

Lipschitz metric on Teichmüller spaces

The Teichmüller space admits a number of natural metrics: the Teichmüller metric, Lipschitz metric, etc.
For any diffeomorphism φ of a surface S and hyperbolic structures σ, τ on S Lipschitz constant $L_\varphi(\sigma, \tau)$ is defined as follows:

$$L_\varphi(\sigma, \tau) = \sup_{x \neq y} \left(\frac{d_\tau(\varphi(x), \varphi(y))}{d_\sigma(x, y)} \right),$$

where d_σ, d_τ are distance functions for structures σ, τ.
The Lipschitz metric d_L:

$$d_L(h, g) = \inf_{\varphi \sim \mathrm{id}} \log \max(L_\varphi(h, g), L_\varphi(g, h))$$

Lipschitz metric for $\mathcal{T}(S^1_{1,0})$

In the case when S is a once-punctured torus $\mathcal{T}(S)$ equipped with the Lipschitz metric is isometric to hyperbolic plane $\mathbb{H}^2$. The corresponding action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$ is the action by linear fractional transformations. Let us consider the generating set $\mathcal{A} = \left\{ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \right\}$ of $\mathrm{SL}(2,\mathbb{Z})$ and for any $x \in \mathbb{H}^2$ consider the map $i_x \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{H}^2$ that sends $g \in \mathrm{SL}(2,\mathbb{Z})$ to $g(x)$. Then i_x is a quasi-isometric embedding of $\mathrm{SL}(2,\mathbb{Z})$ with the metric $\rho_{\mathcal{A}}$ into $\mathbb{H}^2$. Namely there exist $K \geqslant 1$, $C \geqslant 0$ such that for all $g, h \in \mathrm{SL}(2,\mathbb{Z})$

$$\frac{1}{K} d(gx, hx) - C \leqslant \rho_{\mathcal{A}}(g,h) \leqslant K d(gx, hx) - C,$$

where $d(\ ,\ )$ is the hyperbolic metric.

Naturality of the zipped word metric

Theorem (S.)

Let S be an oriented surface without boundary and with non-empty set of punctures, h a hyperbolic structure on S, and $\mathcal{A}$ an admissible generating set of $\mathrm{MCG}(S)$. Let also $i_h \colon \mathrm{MCG}(S) \to \mathcal{T}(S)$ be the map that sends $\varphi \in \mathrm{MCG}(S)$ to the image of h under φ. Then i_h is a quasi-isometric embedding of $\mathrm{MCG}(S)$ equipped with the metric $\rho_{\mathcal{A}}$ into $\mathcal{T}(S)$ equipped with the Lipschitz metric.

Remark

In the case when the surface is a sphere with punctures the similar theorem was obtained by Dynnikov and Wiest.