Weighted inequalities for sublinear integral operators on the semiaxis and applications to analysis on the Lorentz spaces

D.V. Prokhorov[1], V.D. Stepanov[2]

[1]Computing Center of FEB RAS, Khabarovsk, Russia
[2]Peoples Friendship University of Russia, Moscow

The work of the authors was supported by the Russian Scientific Fund (Project 14-11-00443)

International Conference "Geometry Days in Novosibirsk — 2014"
Dedicated to 85th Anniversary of Academician Yuri Grigor'evich Reshetnyak

Novosibirsk, 24–27 September 2014

1. Introduction

Let $\mathbb{R}_+ := [0, \infty)$. Denote $\mathfrak{M}$ the set of all measurable functions on $\mathbb{R}_+$ and $\mathfrak{M}^+ \subset \mathfrak{M}$ the subset of all non-negative functions. If $0 < p \leq \infty$ and $v \in \mathfrak{M}^+$ we define the weighted Lebesgue spaces by

$$L_v^p := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^p} := \left(\int_0^\infty |f(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_v^\infty := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^\infty} := \operatorname*{ess\,sup}_{x \geq 0} v(x)|f(x)| < \infty \right\}.$$

Let $0 < q \le \infty$ and $w \in \mathfrak{M}^+$. We consider sublinear operators on $\mathfrak{M}^+$ of the form

$$(Tf)(x) = \left(\int_x^\infty w(y) \left(\int_0^y k(y,z) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$(\mathscr{T} f)(x) = \left(\int_0^x w(y) \left(\int_y^\infty k(z,y) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$(Sf)(x) = \left(\int_x^\infty w(y) \left(\int_y^\infty k(z,y) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$(\mathscr{S} f)(x) = \left(\int_0^x w(y) \left(\int_0^y k(y,z) f(z) dz \right)^q dy \right)^{\frac{1}{q}}$$

and

$$(\mathbf{T}f)(x) = \left(\int_x^\infty k(y,x)w(y)\left(\int_0^y f(z)dz\right)^q dy\right)^{\frac{1}{q}},$$

$$(\mathfrak{T}f)(x) = \left(\int_0^x k(x,y)w(y)\left(\int_y^\infty f(z)dz\right)^q dy\right)^{\frac{1}{q}},$$

$$(\mathbf{S}f)(x) = \left(\int_x^\infty k(y,x)w(y)\left(\int_y^\infty f(z)dz\right)^q dy\right)^{\frac{1}{q}},$$

$$(\mathfrak{S}f)(x) = \left(\int_0^x k(x,y)w(y)\left(\int_0^y f(z)dz\right)^q dy\right)^{\frac{1}{q}},$$

where $k(x,y) \geq 0$ is a measurable kernel and the right hand sides are to replace by essential supremums

$$(Tf)(x) = \operatorname*{ess\,sup}_{y \geq x} w(y) \textstyle\int_0^y k(y,z)f(z)dz, \tag{1}$$

$$(\mathbf{T}f)(x) = \operatorname*{ess\,sup}_{y \geq x} k(y,x)w(y) \textstyle\int_0^y f(z)dz, \tag{2}$$

and similarly for the others, when $q = \infty$.

We assume that a Borel function $k(x,y) \geq 0$ on $[0,\infty)^2$ satisfies Oinarov's condition: $k(x,y) = 0$ if $x < y$, and there is a constant $D \geq 1$ independent of $x \geq z \geq y \geq 0$ such that

$$\frac{1}{D}\left(k(x,z) + k(z,y)\right) \leq k(x,y) \leq D\left(k(x,z) + k(z,y)\right). \tag{3}$$

Examples: $k(x,y) = (x-y)^\alpha$, $\alpha \geq 0$; $k(x,y) = \log^\beta\left(\frac{x}{y}\right)$, $\beta \geq 0$; $k(x,y) = \left(\int_y^x h(s)ds\right)^\gamma$, $h(s) \geq 0$, $\gamma \geq 0$ and various combinations.

Let $u, v, w \in \mathfrak{M}^+$ be weights, $1 \le p \le \infty$, $0 < r \le \infty$. Our aim is to characterize the weighted inequalities

$$\|Tf\|_{L_u^r} \le C_T \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{4}$$

$$\|\mathscr{T}f\|_{L_u^r} \le C_{\mathscr{T}} \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{5}$$

$$\|Sf\|_{L_u^r} \le C_S \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{6}$$

$$\|\mathscr{S}f\|_{L_u^r} \le C_{\mathscr{S}} \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+ \tag{7}$$

and

$$\|\mathbf{T}f\|_{L_u^r} \le C_{\mathbf{T}} \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{8}$$

$$\|\mathfrak{T}f\|_{L_u^r} \le C_{\mathfrak{T}} \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{9}$$

$$\|\mathbf{S}f\|_{L_u^r} \le C_{\mathbf{S}} \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{10}$$

$$\|\mathfrak{S}f\|_{L_u^r} \le C_{\mathfrak{S}} \|f\|_{L_v^p}, \ f \in \mathfrak{M}^+, \tag{11}$$

where the constants C_T and others are taken as the least possible.

Remark 1. The purpose of the investigation is to prove sharp two-sided estimates of the form

$$c_1(p,q,r,D)F(u,v,w,k) \le C_T \le c_2(p,q,r,D)F(u,v,w,k)$$

for the constants C_T and others, where the multiples $c_i > 0$ depend on parameters of summation and a constant D only and we write

$$C_T \approx F(u,v,w,k).$$

Remark 2. It is known for a *linear integral operator*, say,

$$Kf(x) := W(x) \int_{\mathbb{R}_+} k(x,y)f(y)U(y)dy, \tag{12}$$

acting from L^p into L^q, that if $0 < p < 1, 0 < q < \infty$ and $\|K\|_{L_p \to L_q} < \infty$, then $\|K\|_{L_p \to L_q} = 0$, (Prokhorov-Stepanov, Proc. Steklov Math. Inst. 2003). The same fenomena is valid for the above sublinear operators and, consequently, the interval $0 < p < 1$ is exluded.

Remark 3. The success of a general theory of the last two decades is related to operators (12) with Oinarov's kernel $k(x, y) \geq 0$. First of all it concerns Volterra operators

$$Kf(x) := W(x) \int_0^x k(x, y) f(y) U(y) dy \tag{13}$$

and its dual

$$K^* g(y) := U(y) \int_y^\infty k(x, y) g(y) W(y) dy, \tag{14}$$

If $q = r < \infty$ the inequalities (4)-(11) are reduced to the generalized Hardy-type inequalities with Volterra operators (13) and (14), which were well studied since 90's in various papers and monographs, we list below a small portion:

S. Bloom, R. Kerman, Weighted norm inequalities for operators of Hardy type, *Proc. Amer. Math. Soc.*, **113** (1991), 135-141.

R. Oinarov, Two-sided estimates of the norm of some classes of integral operators, *Proc. Steklov Inst. Math.*, **204** (1994), 205–214.

V.D. Stepanov, Weighted norm inequalities of Hardy type for a class of integral operators,*J. London Math. Soc.*, **50**:1 (1994), 105–120.

Q. Lai, Weighted modular inequalities for Hardy-type operators, *Proc. London Math. Soc.*, **79** (1999), 649–672.

G. Sinnamon and V.D. Stepanov, The weighted Hardy inequality: new proofs and the case p=1. *J. London Math. Soc.* **54** (1996), 89–101.

D.V. Prokhorov, On the boundedness and compactness of a class of integral operators, *J. London Math. Soc.*, **61** (2000), 617–628.

A. Kufner, L.-E. Persson, *Weighted inequalities of Hardy type,* World Scientific Publishing Co., Inc., River Edge, NJ, 2003, xviii+357 pp.

D.V. Prokhorov, On a weighted inequality for a Hardy-type operator, *Proc. Steklov Inst. Math.*, **284** (2014), 216–223.

Remark 4. The cases $p = \infty$ or/and $r = \infty$ have the precise characterization. For instance, for $p = \infty$

$$C_T = \left\| T\left(\frac{1}{v}\right) \right\|_{L^r_u} \tag{15}$$

and for $r = \infty$

$$C_T = \sup_{t \geq 0} U(t) \left\| H_t \right\|_{L^p_v \to L^q_w}, \tag{16}$$

where $U(t) := \operatorname*{ess\,sup}_{0 \leq x \leq t} u(x)$ and $(H_t f)(x) := \chi_{[t,\infty)}(x) \int_0^x k(x,z) f(z) dz$.

2. Operators T and S

Suppose for simplicity that $\int_0^t u < \infty$ for all $t > 0$ and define the functions $\sigma : [0, \infty) \to [0, \infty]$, $\sigma^{-1} : [0, \infty) \to [0, \infty)$ by (here $\inf \varnothing = \infty$)

$$\sigma(x) := \inf \left\{ y > 0 : \int_0^y u \geq 2 \int_0^x u \right\}, \ \ \sigma^{-1}(x) := \inf \left\{ y > 0 : \int_0^y u \geq \frac{1}{2} \int_0^x u \right\}.$$

The functions σ and σ^{-1} are increasing and

$$\int_0^{\sigma(x)} u = 2 \int_0^x u, \ \ \int_0^{\sigma^{-1}(x)} u = \frac{1}{2} \int_0^x u \ \ , x > 0.$$

For $0 < c < d \leq \infty$ and $f \in \mathfrak{M}^+$ we put

$$(H_{c,d} f)(x) := \chi_{[c,d)}(x) \int_{\sigma^{-1}(c)}^x k(x, z) f(z) dz,$$

$$(H_c f)(x) := \chi_{[c,\infty)}(x) \int_0^x k(x, z) f(z) dz.$$

Theorem 1. *Let* $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. *For validity of the inequality* (4) *it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x) \left(\int_x^\infty w\right)^{\frac{r}{q}} \left(\int_0^x k(x,z)f(z)dz\right)^r dx\right)^{\frac{1}{r}} \le A_0 \|f\|_{L_v^p}, \tag{17}$$

$$\left(\int_0^\infty u(x) \left(\int_x^\infty [k(z,x)]^q w(z)dz\right)^{\frac{r}{q}} \left(\int_0^x f\right)^r dx\right)^{\frac{1}{r}} \le A_1 \|f\|_{L_v^p}, \tag{18}$$

if $q < \infty$ *or*

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \ge x} w(y)]^r \left(\int_0^x k(x,z)f(z)dz\right)^r dx\right)^{\frac{1}{r}} \le A_0 \|f\|_{L_v^p}, \tag{19}$$

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \ge x} [w(y)k(y,x)]]^r \left(\int_0^x f\right)^r dx\right)^{\frac{1}{r}} \le A_1 \|f\|_{L_v^p} \tag{20}$$

for $q = \infty$ *hold for all* $f \in \mathfrak{M}^+$ *and the constant*

$$A_2 := \begin{cases} \sup_{t \in (0,\infty)} \left(\int_0^t u\right)^{\frac{1}{r}} \|H_t\|_{L_v^p \to L_w^q}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_0^x u\right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x),\sigma(x)}\|^s_{L_v^p \to L_w^q} dx\right)^{\frac{1}{s}}, & r < p \end{cases} \tag{21}$$

is finite. Moreover, $C_T \approx A_0 + A_1 + A_2$.

Now, for $0 < c < d \leq \infty$ and $f \in \mathfrak{M}^+$ we put

$$(H^*_{c,d}f)(x) := \chi_{[c,d)}(x) \int_x^{\sigma(d)} k(z,x)f(z)dz,$$
$$(H^*_c f)(x) := \chi_{[c,\infty)}(x) \int_x^{\infty} k(z,x)f(z)dz.$$

Theorem 2. *Let* $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. *Then the inequality* (6) *is fulfilled if and only if the inequalities*

$$\left(\int_0^\infty u(x) \left(\int_x^{\sigma^2(x)} w\right)^{\frac{r}{q}} \left(\int_{\sigma^2(x)}^\infty k(z,\sigma^2(x)) f(z) dz\right)^r dx\right)^{1/r} \le \mathbb{A}_0 \|f\|_{L_v^p},$$

$$\left(\int_0^\infty u(x) \left(\int_x^{\sigma^2(x)} [k(\sigma^2(x), z)]^q w(z) dz\right)^{\frac{r}{q}} \left(\int_{\sigma^2(x)}^\infty f\right)^r dx\right)^{1/r} \le \mathbb{A}_1 \|f\|_{L_v^p},$$

if $q < \infty$ *or*

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \in (x,\sigma^2(x))} w(y)]^r \left(\int_{\sigma^2(x)}^\infty k(z,\sigma^2(x)) f(z) dz\right)^r dx\right)^{1/r} \le \mathbb{A}_0 \|f\|_{L_v^p},$$

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \in (x,\sigma^2(x))} [w(y) k(\sigma^2(x), y)]^r \left(\int_{\sigma^2(x)}^\infty f\right)^r dx\right)^{1/r} \le \mathbb{A}_1 \|f\|_{L_v^p}$$

for $q = \infty$ *hold for all* $f \in \mathfrak{M}^+$ *and the constant*

$$\mathbb{A}_2 := \begin{cases} \sup_{t \in (0,\infty)} \left(\int_0^t u\right)^{1/r} \|H_t^*\|_{L_v^p \to L_w^q}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_0^x u\right)^{s/p} \|H_{\sigma^{-1}(x),\sigma(x)}^*\|_{L_v^p \to L_w^q}^s dx\right)^{1/s}, & r < p \end{cases}$$

is finite. Moreover, $C_S \approx \mathbb{A}_0 + \mathbb{A}_1 + \mathbb{A}_2$.

Remark 5. According to the Remark 3 precise characterization of the inequalities (17)-(20), sharp estimates of the norms

$$\|H_t\|_{L^p_v \to L^q_w}, \left\|H_{\sigma^{-1}(x),\sigma(x)}\right\|_{L^p_v \to L^q_w}, \|H^*_t\|_{L^p_v \to L^q_w}, \left\|H^*_{\sigma^{-1}(x),\sigma(x)}\right\|_{L^p_v \to L^q_w}$$

are known and have explicit integral form.

3. Operators $\mathscr{T}$ and $\mathscr{S}$

For finding criteria for (5) and (7) we suppose that $0 < \int_t^\infty u < \infty$ for all $t > 0$ and define the functions $\zeta : [0,\infty) \to [0,\infty)$, $\zeta^{-1} : [0,\infty) \to [0,\infty)$ by

$$\zeta(x) := \sup\left\{y > 0 : \int_y^\infty u \geq \frac{1}{2}\int_x^\infty u\right\},$$

$$\zeta^{-1}(x) := \sup\left\{y > 0 : \int_y^\infty u \geq 2\int_x^\infty u\right\},$$

where $\sup\varnothing = 0$. For $0 \leq c < d < \infty$ and $f \in \mathfrak{M}^+$ we put

$$(\mathscr{H}_{c,d}f)(x) := \chi_{(c,d]}(x)\int_x^{\zeta(d)} k(z,x)f(z)dz,$$

$$(\mathscr{H}_d f)(x) := \chi_{(0,d]}(x)\int_x^\infty k(z,x)f(z)dz,$$

$$(\mathscr{H}^*_{c,d}f)(x) := \chi_{(c,d]}(x)\int_{\zeta^{-1}(c)}^x k(x,z)f(z)dz,$$

$$(\mathscr{H}^*_d f)(x) := \chi_{(0,d]}(x)\int_0^x k(x,z)f(z)dz.$$

Similar to the previous section we prove the following theorems.

Theorem 3. *Let* $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. *For validity of the inequality* (5) *it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x) \left(\int_0^x w\right)^{\frac{r}{q}} \left(\int_x^\infty k(z,x) f(z) dz\right)^r dx\right)^{\frac{1}{r}} \le \mathscr{A}_0 \|f\|_{L_v^p}, \tag{22}$$

$$\left(\int_0^\infty u(x) \left(\int_0^x [k(x,y)]^q w(y) dy\right)^{\frac{r}{q}} \left(\int_x^\infty f\right)^r dx\right)^{\frac{1}{r}} \le \mathscr{A}_1 \|f\|_{L_v^p},$$

if $q < \infty$ *or*

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \in (0,x)} w(y)]^r \left(\int_x^\infty k(z,x) f(z) dz\right)^r dx\right)^{\frac{1}{r}} \le \mathscr{A}_0 \|f\|_{L_v^p},$$

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \in (0,x)} [w(y) k(x,y)]]^r \left(\int_x^\infty f\right)^r dx\right)^{\frac{1}{r}} \le \mathscr{A}_1 \|f\|_{L_v^p}$$

for $q = \infty$ *hold for all* $f \in \mathfrak{M}^+$ *and the constant*

$$\mathscr{A}_2 := \begin{cases} \sup_{t \in (0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{r}} \|\mathscr{H}_t\|_{L_v^p \to L_w^q}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u\right)^{\frac{s}{p}} \|\mathscr{H}_{\zeta^{-1}(x),\zeta(x)}\|_{L_v^p \to L_w^q}^s dx\right)^{\frac{1}{s}}, & r < p, \end{cases} \tag{23}$$

is finite. Moreover, $C_{\mathscr{T}} \approx \mathscr{A}_0 + \mathscr{A}_1 + \mathscr{A}_2$.

Theorem 4. *Let* $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. *For validity of the inequality* (7) *it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x) \left(\int_{\zeta^{-2}(x)}^x w\right)^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} k(\zeta^{-2}(x), z) f(z) dz\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{A}_0 \|f\|_{L_v^p},$$

$$\left(\int_0^\infty u(x) \left(\int_{\zeta^{-2}(x)}^x w(y)[k(y, \zeta^{-2}(x))]^q dy\right)^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} f\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{A}_1 \|f\|_{L_v^p},$$

if $q < \infty$ *or*

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \in (\zeta^{-2}(x), x)} w(y)]^r \left(\int_0^{\zeta^{-2}(x)} k(\zeta^{-2}(x), z) f(z) dz\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{A}_0 \|f\|_{L_v^p},$$

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \in (\zeta^{-2}(x), x)} [w(y) k(y, \zeta^{-2}(x))]]^r \left(\int_0^{\zeta^{-2}(x)} f\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{A}_1 \|f\|_{L_v^p}$$

for $q = \infty$ *hold for all* $f \in \mathfrak{M}^+$ *and the constant*

$$\mathbf{A}_2 := \begin{cases} \sup_{t \in (0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{r}} \|\mathscr{H}_t^*\|_{L_v^p \to L_w^q}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u\right)^{\frac{s}{p}} \|\mathscr{H}_{\zeta^{-1}(x), \zeta(x)}^*\|_{L_v^p \to L_w^q}^s dx\right)^{\frac{1}{s}}, & r < p \end{cases} \tag{24}$$

is finite. Moreover, $C_{\mathscr{S}} \approx \mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2$.

4. Operators $\mathbf{T}$ and $\mathbf{S}$

Let the functions σ and σ^{-1} be the same as in the Section 2. For $0 < c < d \leq \infty$ and $f \in \mathfrak{M}^+$ we put

$$(\mathbf{H}_{c,d}f)(x) := \chi_{[c,d)}(x) \int_{\sigma^{-1}(c)}^{x} f(z)dz, \quad (\mathbf{H}_c f)(x) := \chi_{[c,\infty)}(x) \int_0^x f(z)dz,$$

$$(\mathbf{H}^*_{c,d}f)(x) := \chi_{[c,d)}(x) \int_x^{\sigma(d)} f(z)dz, \quad (\mathbf{H}^*_c f)(x) := \chi_{[c,\infty)}(x) \int_x^{\infty} f(z)dz.$$

Theorem 5. *Let $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. For validity of* (8) *it is necessary and sufficient that $B := B_0 + B_1 + B_2 < \infty$, where B_0 and B_1 are the least possible constants in the inequalities*

$$\left(\int_0^\infty u(x) \left(\int_x^\infty k(y,x) w(y) dy\right)^{\frac{r}{q}} \left(\int_0^x f\right)^r dx\right)^{\frac{1}{r}} \le B_0 \|f\|_{L^p_v}, \tag{25}$$

$$\left(\int_0^\infty u(x) [k(\sigma^2(x), x)]^{\frac{r}{q}} \left(\int_{\sigma^2(x)}^\infty w(y) \left(\int_0^y f\right)^q dy\right)^{\frac{r}{q}} dx\right)^{\frac{1}{r}} \le B_1 \|f\|_{L^p_v}, \tag{26}$$

if $q < \infty$ or

$$\left(\int_0^\infty u(x) [\operatorname*{ess\,sup}_{y \ge x} k(y,x) w(y)]^r \left(\int_0^x f\right)^r dx\right)^{\frac{1}{r}} \le B_0 \|f\|_{L^p_v}, \tag{27}$$

$$\left(\int_0^\infty u(x) [k(\sigma^2(x), x)]^r \left(\operatorname*{ess\,sup}_{y \ge \sigma^2(x)} w(y) \int_0^y f\right)^r dx\right)^{\frac{1}{r}} \le B_1 \|f\|_{L^p_v}, \tag{28}$$

for $q = \infty$ and B_2 is defined by

$$B_2 := \begin{cases} \sup_{t>0} \left(\int_0^t u\right)^{\frac{1}{r}} \|\mathbf{H}_t\|_{L^p_v \to L^q_{w(\cdot)k(\cdot,t)}}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_0^x u\right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(x), \sigma^2(x)}\|^s_{L^p_v \to L^q_{w(\cdot)k(\cdot,\sigma^{-1}(x))}} dx\right)^{\frac{1}{s}}, & r < p. \end{cases}$$

Moreover, $C_{\mathbf{T}} \approx B$.

Theorem 6. *Let* $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. *For validity of* (10) *it is necessary and sufficient that* $\mathbb{B} := \mathbb{B}_0 + \mathbb{B}_1 + \mathbb{B}_2 < \infty$, *where* $\mathbb{B}_0$ *and* $\mathbb{B}_1$ *are the least possible constants in the inequalities*

$$\left(\int_0^\infty u(x)\left(\int_x^{\sigma^3(x)} k(y,x)w(y)dy\right)^{\frac{r}{q}}\left(\int_{\sigma^3(x)}^\infty f\right)^r dx\right)^{1/r} \le \mathbb{B}_0\|f\|_{L_v^p}, \qquad (29)$$

$$\left(\int_0^\infty u(x)k(\sigma^2(x),x)^{\frac{r}{q}}\left(\int_{\sigma^2(x)}^\infty w(y)\left(\int_y^\infty f\right)^q dy\right)^{\frac{r}{q}} dx\right)^{1/r} \le \mathbb{B}_1\|f\|_{L_v^p}, \qquad (30)$$

when $q < \infty$ *and*

$$\left(\int_0^\infty u(x)[\operatorname*{ess\,sup}_{x\le y\le\sigma^3(x)} k(y,x)w(y)]^r\left(\int_{\sigma^3(x)}^\infty f\right)^r dx\right)^{1/r} \le \mathbb{B}_0\|f\|_{L_v^p}, \qquad (31)$$

$$\left(\int_0^\infty u(x)[k(\sigma^2(x),x)]^r\left(\operatorname*{ess\,sup}_{y\ge\sigma^2(x)} w(y)\int_y^\infty f\right)^r dx\right)^{1/r} \le \mathbb{B}_1\|f\|_{L_v^p}, \qquad (32)$$

if $q = \infty$. *The constant* $\mathbb{B}_2$ *is given by*

$$\mathbb{B}_2 := \begin{cases} \sup_{t>0}\left(\int_0^t u\right)^{\frac{1}{r}}\|\mathbf{H}_t^*\|_{L_v^p\to L^q_{w(\cdot)k(\cdot,t)}}, & p \le r, \\ \left(\int_0^\infty u(x)\left(\int_0^x u\right)^{\frac{s}{p}}\|\mathbf{H}^*_{\sigma^{-1}(x),\sigma^2(x)}\|^s_{L_v^p\to L^q_{w(\cdot)k(\cdot,\sigma^{-1}(x))}} dx\right)^{\frac{1}{s}}, & r < p. \end{cases}$$

Moreover, $C_{\mathbf{S}} \approx \mathbb{B}$.

5. Operators $\mathfrak{T}$ and $\mathfrak{S}$

Let the functions $\zeta, \zeta^{-1} : [0,\infty) \to [0,\infty)$ be the same as in the Section 3. For $0 \le c < d < \infty$ and $f \in \mathfrak{M}^+$ we define operators

$$
\begin{aligned}
(\mathfrak{H}_{c,d} f)(x) &:= \chi_{(c,d]}(x) \int_x^{\zeta(d)} f(z)dz, \\
(\mathfrak{H}_{d} f)(x) &:= \chi_{(0,d]}(x) \int_x^{\infty} f(z)dz, \\
(\mathfrak{H}^*_{c,d} f)(x) &:= \chi_{(c,d]}(x) \int_{\zeta^{-1}(c)}^{x} f(z)dz, \\
(\mathfrak{H}^*_{d} f)(x) &:= \chi_{(0,d]}(x) \int_0^{x} f(z)dz.
\end{aligned}
$$

The following theorems are true.

Theorem 7. *Let* $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. *For validity of the inequality* (9) *it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x) \left(\int_0^x k(x,y)w(y)dy\right)^{\frac{r}{q}} \left(\int_x^\infty f\right)^r dx\right)^{\frac{1}{r}} \le \mathscr{B}_0 \|f\|_{L^p_v},$$

$$\left(\int_0^\infty u(x)[k(\zeta^{-2}(x),x)]^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} w(y) \left(\int_y^\infty f\right)^q dy\right)^{\frac{r}{q}} dx\right)^{\frac{1}{r}} \le \mathscr{B}_1 \|f\|_{L^p_v},$$

if $q < \infty$ *or*

$$\left(\int_0^\infty u(x)[\operatorname*{ess\,sup}_{y\in(0,x)} k(x,y)w(y)]^r \left(\int_x^\infty f\right)^r dx\right)^{\frac{1}{r}} \le \mathscr{B}_0 \|f\|_{L^p_v},$$

$$\left(\int_0^\infty u(x)[k(\zeta^{-2}(x),x)]^r \left(\operatorname*{ess\,sup}_{y\in(0,\zeta^{-2}(x))} w(y) \int_y^\infty f\right)^r dx\right)^{\frac{1}{r}} \le \mathscr{B}_1 \|f\|_{L^p_v}$$

for $q = \infty$ *hold for all* $f \in \mathfrak{M}^+$ *and the constant*

$$\mathscr{B}_2 := \begin{cases} \sup_{t\in(0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{r}} \|\mathfrak{H}_t\|_{L^p_v \to L^q_w}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u\right)^{\frac{s}{p}} \|\mathfrak{H}_{\zeta^{-1}(x),\zeta^2(x)}\|^s_{L^p_v \to L^q_{w(\cdot)k(\zeta^{-1}(x),\cdot)}} dx\right)^{\frac{1}{s}}, & r < p, \end{cases}$$

is finite. Moreover, $C_{\mathfrak{T}} \approx \mathscr{B}_0 + \mathscr{B}_1 + \mathscr{B}_2$.

Theorem 8. *Let $1 \le p < \infty$, $0 < r < \infty$, $0 < q \le \infty$, $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$. For validity of the inequality* (11) *it is necessary and sufficient that the inequalities*

$$\left(\int_0^\infty u(x) \left(\int_{\zeta^{-3}(x)}^x k(x,y)w(y)dy\right)^{\frac{r}{q}} \left(\int_0^{\zeta^{-3}(x)} f\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_0 \|f\|_{L^p_v},$$

$$\left(\int_0^\infty u(x)[k(\zeta^{-2}(x),x)]^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} w(y) \left(\int_0^{\zeta^{-2}(x)} f\right)^q dy\right)^{\frac{r}{q}} dx\right)^{\frac{1}{r}} \le \mathbf{B}_1 \|f\|_{L^p_v},$$

if $q < \infty$ or

$$\left(\int_0^\infty u(x)[\operatorname*{ess\,sup}_{y \in (\zeta^{-3}(x),x)} k(x,y)w(y)]^r \left(\int_0^{\zeta^{-3}(x)} f\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_0 \|f\|_{L^p_v},$$

$$\left(\int_0^\infty u(x)[k(\zeta^{-2}(x),x)]^r \left(\operatorname*{ess\,sup}_{y \in (0,\zeta^{-2}(x))} w(y) \int_0^y f\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_1 \|f\|_{L^p_v}$$

for $q = \infty$ hold for all $f \in \mathfrak{M}^+$ and the constant

$$\mathbf{B}_2 := \begin{cases} \sup_{t \in (0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{r}} \|\mathfrak{H}^*_t\|_{L^p_v \to L^q_{w(\cdot)k(t,\cdot)}}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u\right)^{\frac{s}{p}} \|\mathfrak{H}^*_{\zeta^{-1}(x),\zeta^2(x)}\|^s_{L^p_v \to L^q_{w(\cdot)k(\zeta^2(x),\cdot)}} dx\right)^{\frac{1}{s}}, & r < p \end{cases}$$

is finite. Moreover, $C_{\mathfrak{S}} \approx \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2$.

6. Boundedness of classical operators in the Lorentz spaces

Since 90's $\Lambda-$*analysis* was started, – problems related to characterization of classical operators in the Lorentz spaces. Remind, that given f on $\mathbb{R}^n$, the non-increasing rearrangement f^* is defined by

$$f^*(t) := \inf\{s > 0 : \mathrm{mes}\{x \in \mathbb{R}^n : |f(x)| > s\} \le t\}, \ \ t \in \mathbb{R}_+.$$

Given $p \in (0, \infty)$ and a weght $v(x) \ge 0$ on $\mathbb{R}_+$ *Lorentz* Λ*-space* $\Lambda^p(v)$ is a collection of all functions f such, that $\|f\|_{\Lambda^p(v)} < \infty$, where

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty [f^*(t)]^p v(t)dt\right)^{1/p}.$$

A problem is to characterize the boundedness $T : \Lambda^p(v) \to \Lambda^q(w)$. Let

$$L^{\downarrow}_{p,v} := \left\{f \in \mathfrak{M}^{\downarrow} : \|f\|_{p,v} := \left(\int_{\mathbb{R}_+} |f(x)|^p \, v(x)dx\right)^{1/p} < \infty\right\}.$$

If for some $T : \Lambda^p(v) \to \Lambda^q(w)$ the inequality

$$[Tf]^*(t) \ll [Sf^*](t), \, t > 0$$

holds, where S is a positive operator, then

$$S : L^{\downarrow}_{p,v} \to L^{+}_{q,w} \Rightarrow T : \Lambda^p(v) \to \Lambda^q(w)$$

and

$$\|T\|_{\Lambda^p(v)\to\Lambda^q(w)} \ll \|S\|_{L^{\downarrow}_{p,v}\to L^{+}_{q,w}}.$$

If

$$[Tf]^*(t) \approx \int_0^\infty k(t,s)u(s)f^*(s)ds, \tag{33}$$

where $k(t,s) \geq 0$ and $u(s) \geq 0$, then the boundedness of $T : \Lambda^p(v) \to \Lambda^q(w)$ is equivalent to the validity of

$$\left(\int_0^\infty \left(\int_0^\infty k(t,s)f(s)u(s)ds\right)^q w(t)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty [f(t)]^p v(t)dt\right)^{\frac{1}{p}} \tag{34}$$

for all $f \in \mathfrak{M}^\downarrow$.

Examples. The Hardy-Littlewood maximal operator

$$Mf(x) := \sup_{B \ni x} \frac{1}{\mathrm{mes} B} \int_B |f(y)|dy$$

has the two-sided estimate

$$[Mf]^*(t) \approx \frac{1}{t}\int_0^t f^*(s)ds.$$

Consequently, the boundedness $M : \Lambda^p(v) \to \Lambda^q(u)$ is equivalent to

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)ds\right)^q w(t)dt\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty [f(t)]^p v(t)dt\right)^{\frac{1}{p}}, \; f \in \mathfrak{M}^\downarrow. \tag{35}$$

Analogously, for the Riesz potential

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}},\ 0<\alpha<n$$

and the Hilbert transform

$$\mathscr{H} f(x) := \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{f(x-y)dy}{y}$$

the estimates

$$(I_\alpha f)^*(t) \ll \left[t^{\frac{\alpha}{n}-1}\int_0^t f^*(z)dz + \int_t^\infty t^{\frac{\alpha}{n}-1} f^*(z)dz\right] \ll (I_\alpha \widetilde{f})^*(t),\ t>0,$$

where $\widetilde{f}(y) = f^*(A|y|^n)$ and

$$(\mathscr{H} f)^*(t) \ll C_1\left[t^{-1}\int_0^t f^*(z)dz + \int_t^\infty z^{-1} f^*(z)dz\right] \ll (\mathscr{H} f^*)^*(t),$$

hold. For $\gamma\in(0,n)$, the fractional maximal operator M_γ is given by

$$M_\gamma f(x) := \sup_{B\ni x} |B|^{\frac{\gamma}{n}-1}\int_B |f(y)|dy.$$

Then

$$\|M_\gamma\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \|S_\gamma\|_{L^\downarrow_{p,v}\to L^+_{q,w}},$$

where

$$S_\gamma f(t) := \sup_{\tau\ge t} \tau^{\frac{\gamma}{n}-1}\int_0^\tau f^*(s)ds$$

Weighted Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f(y)u(y)dy\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \; f \in \mathfrak{M}^\downarrow, \qquad (36)$$

has numerous applications. The charcterization of (36) was taken about 15 years.

M. Ariño M, B. Muckenhoupt, Maximal funcrions on classical Lorentz spaces and Hardy's inequality with weights for non-increasing functions, *Trans. Amer. Math. Soc.*, **320** (1990), 727–735.
E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.*, **96** (1990), 145–158.
V.D. Stepanov, The weighted Hardy's inequality for nonincreasing functions, *Trans. Amer. Math. Soc.*, **338** (1993), 173–186.
M.L. Goldman, Sharp estimates for the norms of Hardy-type operators on cones of quasimonotone functions, *Proc. Steklov Inst. Math.*, **232** (2001), 109–137.
G. Bennett, K.-G. Grosse-Erdmann, Weighted Hardy inequality for decreasing sequences and functions, *Math. Ann.*, **334** (2006), 489–531.
with latest survey
A. Gogatishvili, V.D. Stepanov, Reduction theorems for weighted integral inequalities on the cone of monotone functions, *Russian Math. Surveys*, **68(4)** (2013), 597–664.

Put $U(t) := \int_0^t u, V(t) := \int_0^t v, W(t) := \int_0^t w$.

Theorem 9. *For the least possible constant C in the inequality* (36) *the following equivalences hold:*

(a) *Let $1 < p \le q < \infty$, then $C \approx A_0 + A_1$, where*

$$A_0 = \sup_{t>0} A_0(t) := \sup_{t>0} \left(\int_0^t U^q w \right)^{\frac{1}{q}} V^{-\frac{1}{p}}(t),$$

and

$$A_1 := \sup_{t>0} \left(\int_t^\infty w \right)^{\frac{1}{q}} \left(\int_0^t \left(\frac{U}{V} \right)^{p'} v \right)^{\frac{1}{p'}}.$$

(b) *If $0 < q < p < \infty, 1 < p < \infty$, then $C \approx B_0 + B_1$, where*

$$B_0 = B_0(p,q) := \left(\int_0^\infty V^{-\frac{r}{p}}(t) \left(\int_0^t U^q w \right)^{\frac{r}{p}} U^q(t) w(t) dt \right)^{\frac{1}{r}},$$

and

$$B_1 = B_1(p,q) := \left(\int_0^\infty \left(\int_t^\infty w \right)^{\frac{r}{p}} \left(\int_0^t \left(\frac{U}{V} \right)^{p'} v \right)^{\frac{r}{p'}} w(t) dt \right)^{\frac{1}{r}}.$$

(c) *For* $0 < q < p \le 1$, $C \approx B_0 + \mathscr{B}_1$, *where*

$$\mathscr{B}_1 = \mathscr{B}_1(p,q) := \left(\int_0^\infty \left(\operatorname*{esssup}_{s\in[0,t]} \frac{U^p(s)}{V(s)}\right)^{\frac{r}{p}} \left(\int_t^\infty w\right)^{\frac{r}{p}} w(t)dt\right)^{\frac{1}{r}}.$$

(d) *If* $0 < p \le q < \infty,\ 0 < p \le 1,$ *then* $C = \mathscr{A}_1$,

$$\mathscr{A}_1 := \sup_{t>0} V^{-\frac{1}{p}}(t) \left(\int_0^\infty U^q(\min\{s,t\}) w(s)ds\right)^{\frac{1}{q}}.$$

It is well known that the set of functions $\mathfrak{M}_h^{\downarrow} := \{\int_x^\infty h, h \in L_1 \cap \mathfrak{M}^+\}$ is dense in $\mathfrak{M}^{\downarrow}$. Consequently, if we change $[f(x)]^p = \int_x^\infty h$ in (36), we obtain an equivalent inequality

$$\left(\int_0^\infty \left(\int_0^x \left(\int_y^\infty h\right)^{\frac{1}{p}} u(y)dy\right)^q w(x)dx\right)^{\frac{p}{q}} \le C^p \int_0^\infty hV,\ h \in \mathfrak{M}^+, \quad (37)$$

characterized by Theorem 3.
Analogously, we solve the inequalities with the Riesz potential, Hilbert transform, fractional Hardy-Littlewood operator, etc.

7. $\Gamma^p(v) \to \Gamma^q(w)$ boundedness of the maximal operator

The Lorentz $\Gamma-$spaces were introduced by E.T. Sawyer while working on characterization of the boundedness of classical operators in the weighted Lorentz $\Lambda-$spaces. More exactly, if $v \in \mathfrak{M}^+$ and $0 < p < \infty$, then

$$\Gamma^p(v) = \left\{ f \text{ measurable on } \mathbb{R}^n : \left(\int_0^\infty [f^{**}(x)]^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\},$$

where $f^{**}(x) := \frac{1}{x} \int_0^x f^*(t)dt$. Certainly, the natural problem was to characterize

$$M : \Lambda^p(v) \to \Gamma^q(u), M : \Gamma^p(v) \to \Lambda^q(u), M : \Gamma^p(v) \to \Gamma^q(u).$$

Using the relation $[Mf]^*(x) \approx \frac{1}{x} \int_0^x f^*$, we see that $M : \Gamma^p(v) \to \Gamma^q(u)$ boundedness is equivalent to the weighted inequality

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x \left(\frac{1}{y} \int_0^y f \right) dy \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty \left(\frac{1}{t} \int_0^t f \right)^p v(t) dt \right)^{\frac{1}{p}}, \tag{38}$$

restricted on the cone $f \in \mathfrak{M}^\downarrow$. Moreover, the least possible constant C is equivalent to the norm of M

$$C \approx \|M\|_{\Gamma^p(v) \to \Gamma^q(u)} := \sup_{0 \neq f \in \Gamma^p(v)} \frac{\|Mf\|_{\Gamma^q(u)}}{\|f\|_{\Gamma^p(v)}}.$$

The inequality (38) was first characterized in the diagonal case $1 < p = q < \infty, u = v$.

V.D. Stepanov, Integral operators on the cone of monotone functions. *J. London Math. Soc.* **48** (1993), 465–487.

For $1 < p, q < \infty, u \neq v$ in

M.L. Goldman, H.P. Heinig and V.D. Stepanov, On the principle of duality in Lorentz spaces, *Canad. J. Math.* **48** (1996), 959–979.

and

G. Sinnamon, Embeddings of concave functions and duals of Lorentz spaces, *Publ. Mat.* **46** (2002), 489–515.

The case $0 < p, q < 1$ is new.

Applying Theorems 3 and 4 we solve the problem for all $0 < p, q < \infty$ and our criteria have an explicit integral form.

Let $\Omega_{1,0} := \{g \in \mathfrak{M}^{\downarrow}, tg(t) \in \mathfrak{M}^{\uparrow}\}$. Then $F(t) = \frac{1}{t}\int_0^t f \in \Omega_{1,0}$ for any $f \in \mathfrak{M}^{\downarrow}$ and $F^p \in \Omega_{p,0} := \{g \in \mathfrak{M}^{\downarrow}, t^p g(t) \in \mathfrak{M}^{\uparrow}\}$. By the change $G = F^p$ (38) becomes equivalent to

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x G^{\frac{1}{p}}\right)^q u(x)dx\right)^{\frac{p}{q}} \le C^p \int_0^\infty Gv, \ G \in \Omega_{p,0} \tag{39}$$

and using the density of functions $\int_0^\infty \frac{h(z)dz}{y^p+z^p}$ in the set $\Omega_{p,0}$ we reduce (39) to the inequality

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x \left(\int_0^\infty \frac{h(z)dz}{y^p+z^p}\right)^{\frac{1}{p}} dy\right)^q u(x)dx\right)^{\frac{p}{q}} \lesssim C^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{40}$$

where

$$V(z) = \int_0^\infty \frac{v(y)dy}{y^p+z^p}.$$

Since

$$\int_0^\infty \frac{h(z)dz}{y^p + z^p} \approx \int_y^\infty \frac{h(z)dz}{z^p} + \frac{1}{y^p}\int_0^y h(z)dz,$$

(40) is characterized by the following pair of inequalities:

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x \left(\int_y^\infty h(z)dz\right)^{\frac{1}{p}} dy\right)^q u(x)dx\right)^{\frac{p}{q}} \le C_1^p \int_0^\infty h(t)t^p V(t)dt,\ h \in \mathfrak{M}^+$$

and

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x \left(\int_0^y h(z)dz\right)^{\frac{1}{p}} \frac{dy}{y}\right)^q u(x)dx\right)^{\frac{p}{q}} \le C_2^p \int_0^\infty hV,\ h \in \mathfrak{M}^+,$$

which are of the form (5) and (7), respectively. Moreover,

$$C \approx C_1 + C_2.$$

Hence, applying Theorems 3.1 and 3.2, we see that and

$$C_1 \approx \mathscr{A}_0 + \mathscr{A}_2 \tag{41}$$

and

$$C_2 \approx \mathbf{A}_0 + \mathbf{A}_2, \tag{42}$$

where the constants A's are defined by (22) and (23) for (41) and by (18) and (24) for (42) under related changes of weights.

Suppose for simplicity that $0 < \int_t^\infty s^{-q}u(s)ds < \infty$ for all $t > 0$. Now, the functions ζ and ζ^{-1} are defined by

$$\zeta(x) := \sup\left\{y > 0 : \int_y^\infty s^{-q}u(s)ds \geq \frac{1}{2}\int_x^\infty s^{-q}u(s)ds\right\},$$

$$\zeta^{-1}(x) := \sup\left\{y > 0 : \int_y^\infty s^{-q}u(s)ds \geq 2\int_x^\infty s^{-q}u(s)ds\right\}.$$

For $0 \leq c < d < \infty$ and $h \in \mathfrak{M}^+$ we put

$$\begin{aligned}
(\mathscr{H}_{c,d}h)(x) &:= \chi_{(c,d]}(x)\int_x^{\zeta(d)} h,\\
(\mathscr{H}_{d}h)(x) &:= \chi_{(0,d]}(x)\int_x^{\infty} h,\\
(\mathscr{H}^*_{c,d}h)(x) &:= \chi_{(c,d]}(x)\int_{\zeta^{-1}(c)}^{x} h,\\
(\mathscr{H}^*_{d}h)(x) &:= \chi_{(0,d]}(x)\int_0^{x} h.
\end{aligned}$$

By Theorem 3 $\mathscr{A}_0$ is the least possible constant in the inequaity

$$\left(\int_0^\infty u(x)\left(\int_x^\infty h\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}} \le \mathscr{A}_0^p \int_0^\infty h(z)z^pV(z)dz,\ h \in \mathfrak{M}^+$$

and $\mathscr{A}_2$ is defined by

$$\mathscr{A}_2^p :=$$

$$\begin{cases} \sup_{t\in(0,\infty)}\left(\int_t^\infty s^{-q}u(s)ds\right)^{\frac{p}{q}} \|\mathscr{H}_t\|_{L^1_{z^pV(z)}\to L^{\frac{1}{p}}}, & p \le q, \\ \left(\int_0^\infty x^{-q}u(x)\left(\int_x^\infty s^{-q}u(s)ds\right)^{\frac{q}{p-q}} \|\mathscr{H}_{\zeta^{-1}(x),\zeta(x)}\|^{\frac{q}{p-q}}_{L^1_{z^pV(z)}\to L^{\frac{1}{p}}}dx\right)^{\frac{p-q}{q}}, & q < p. \end{cases}$$

Also, by Theorem 4 $\mathbf{A}_0$ is the best possible constant in the inequaity

$$\left(\int_0^\infty x^{-q}u(x)\left(\log\frac{x}{\zeta^{-2}(x)}\right)^q\left(\int_0^{\zeta^{-2}(x)}h\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}}\le \mathbf{A}_0^p\int_0^\infty hV,\ h\in\mathfrak{M}^+$$

and $\mathbf{A}_2$ is determined from

$$\mathbf{A}_2^p := \begin{cases} \sup_{t\in(0,\infty)}\left(\int_t^\infty s^{-q}u(s)ds\right)^{\frac{p}{q}}\|\mathscr{H}_t^*\|_{L_V^1\to L_{\frac{1}{y}}^{\frac{1}{p}}}, & p\le q,\\ \left(\int_0^\infty x^{-q}u(x)\left(\int_x^\infty s^{-q}u(s)ds\right)^{\frac{q}{p-q}}\|\mathscr{H}_{\zeta^{-1}(x),\zeta(x)}^*\|_{L_V^1\to L_{\frac{1}{y}}^{\frac{1}{p}}}^{\frac{q}{p-q}}dx\right)^{\frac{p-q}{q}}, & q<p. \end{cases}$$

By well known results we have

$$\mathscr{A}_0^p = \sup_{t>0} \frac{\left(\int_0^t u\right)^{\frac{p}{q}}}{t^p V(t)},\ p \le q$$

and

$$\mathscr{A}_0^p \approx \left(\int_0^\infty [t^p V(t)]^{\frac{q}{q-p}} \left(\int_0^t u \right)^{\frac{q}{p-q}} u(t) dt \right)^{\frac{p-q}{q}},\ q < p.$$

Analogously, we find

$$\mathbf{A}_0^p = \sup_{t>0} \frac{\left(\int_{\zeta^2(t)}^\infty x^{-q} u(x) \left(\log \frac{x}{\zeta^{-2}(x)} \right)^q \right)^{\frac{p}{q}}}{V(t)},\ p \le q$$

and for $q < p$

$$\mathbf{A}_0^p \approx \left(\int_0^\infty \left(\frac{\int_x^\infty s^{-q} u(s) \left(\log \frac{s}{\zeta^{-2}(s)} \right)^q ds}{V(\zeta^{-2}(x))} \right)^{\frac{q}{p-q}} x^{-q} u(x) \left(\log \frac{x}{\zeta^{-2}(x)} \right)^q dx \right)^{\frac{p-q}{q}}$$

Similarly, we obtain

$$\|\mathscr{H}_t\|_{L^1_{z^p V(z)} \to L^{\frac{1}{p}}} = [V(t)]^{-1},\ 0 < p \le 1$$

and

$$\|\mathscr{H}_t\|_{L^1_{z^p V(z)} \to L^{\frac{1}{p}}} \approx \left(\int_0^t [V(x)]^{\frac{1}{1-p}} \frac{dx}{x} \right)^{p-1},\ p > 1,$$

so that it follows from (36) for $p \le q$

$$\mathscr{A}_2 = \sup_{t \in (0,\infty)} \left(\int_t^\infty s^{-q} u(s) ds \right)^{\frac{1}{q}} [V(t)]^{-\frac{1}{p}},\ 0 < p \le 1,$$

and

$$\mathscr{A}_2 \approx \sup_{t \in (0,\infty)} \left(\int_t^\infty s^{-q} u(s) ds \right)^{\frac{1}{q}} \left(\int_0^t [V(x)]^{\frac{1}{1-p}} \frac{dx}{x} \right)^{\frac{1}{p'}}, p > 1,$$

where $p' := \frac{p}{p-1}$.

By the same way,

$$\left\|\mathscr{H}_{\zeta^{-1}(x),\zeta(x)}\right\|_{L^1_{z^pV(z)}\to L^{\frac{1}{p}}} = \left[\frac{\zeta(x)-\zeta^{-1}(x)}{\zeta(x)}\right]^p \frac{1}{V(\zeta(x))},\ 0<p\le 1$$

and

$$\left\|\mathscr{H}_{\zeta^{-1}(x),\zeta(x)}\right\|_{L^1_{z^pV(z)}\to L^{\frac{1}{p}}} \approx \left(\int_{\zeta^{-1}(x)}^{\zeta(x)}[t^pV(t)]^{\frac{1}{1-p}}(t-\zeta^{-1}(x))^{\frac{1}{p-1}}dt\right)^{p-1},\ p>1.$$

Hence, from (36) we see that for $q<p$

$$\mathscr{A}_2 \approx \left(\int_0^\infty x^{-q}u(x)\left(\int_x^\infty s^{-q}u(s)ds\right)^{\frac{q}{p-q}}\left[\frac{(\zeta(x)-\zeta^{-1}(x)))}{\zeta(x)[V(\zeta(x))]^{\frac{1}{p}}}\right]^{\frac{pq}{p-q}}dx\right)^{\frac{p-q}{pq}},$$

if $0<p\le 1$ and when $p>1$

$$\mathscr{A}_2 \approx$$

$$\left(\int_0^\infty x^{-q}u(x)\left(\int_x^\infty s^{-q}u(s)ds\right)^{\frac{q}{p-q}}\left(\int_{\zeta^{-1}(x)}^{\zeta(x)}\left(\frac{t-\zeta^{-1}(x)}{t^pV(t)}\right)^{\frac{1}{p-1}}dt\right)^{\frac{q(p-1)}{p-q}}dx\right)^{\frac{p-q}{pq}}$$

Similarly,

$$\|\mathscr{H}_t^*\|_{L^1_V \to L^{\frac{1}{p}}_{\frac{1}{y}}} = \sup_{s\in(0,t)} [V(s)]^{-1} \left(\log \frac{t}{s}\right)^p, \ 0 < p \le 1$$

and

$$\|\mathscr{H}_t^*\|_{L^1_V \to L^{\frac{1}{p}}_{\frac{1}{y}}} \approx \left(\int_0^t [V(x)]^{\frac{1}{1-p}} \left(\log \frac{t}{s}\right)^{\frac{1}{p-1}} \frac{dx}{x}\right)^{p-1}, \ p > 1.$$

Now, it follows from (37) for $p \le q$

$$\mathbf{A}_2 = \sup_{t\in(0,\infty)} \left(\int_t^\infty s^{-q} u(s) ds\right)^{\frac{1}{q}} \sup_{s\in(0,t)} [V(s)]^{-\frac{1}{p}} \log \frac{t}{s}, \ 0 < p \le 1$$

and

$$\mathbf{A}_2 \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty s^{-q} u(s) ds\right)^{\frac{1}{q}} \left(\int_0^t [V(x)]^{\frac{1}{1-p}} \left(\log \frac{t}{s}\right)^{\frac{1}{p-1}} \frac{dx}{x}\right)^{\frac{1}{p'}}, p > 1.$$

We have

$$\left\|\mathscr{H}^*_{\zeta^{-1}(x),\zeta(x)}\right\|_{L^1_V\to L^{\frac{1}{p}}_{\frac{1}{y}}} = \sup_{s\in(\zeta^{-1}(x),\zeta(x))} \frac{\left(\log\frac{\zeta(x)}{s}\right)^p}{V(s)},\ 0<p\le 1$$

and

$$\left\|\mathscr{H}^*_{\zeta^{-1}(x),\zeta(x)}\right\|_{L^1_V\to L^{\frac{1}{p}}_{\frac{1}{y}}} \approx \left(\int_{\zeta^{-1}(x)}^{\zeta(x)} [V(t)]^{\frac{1}{1-p}} \left(\log\frac{\zeta(x)}{t}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right)^{p-1},\ p>1.$$

Thus, from (37) we find for $q<p$

$$\mathbf{A}_2 \approx \left(\int_0^\infty x^{-q}u(x)\left(\int_x^\infty s^{-q}u(s)ds\right)^{\frac{q}{p-q}}\left[\sup_{s\in(\zeta^{-1}(x),\zeta(x))}\frac{\left(\log\frac{\zeta(x)}{s}\right)^p}{V(s)}\right]^{\frac{q}{p-q}}dx\right)^{\frac{p-q}{pq}},$$

if $0<p\le 1$ and when $p>1$.

$$\mathbf{A}_2 \approx$$

$$\left(\int_0^\infty x^{-q}u(x)\left(\int_x^\infty s^{-q}u(s)ds\right)^{\frac{q}{p-q}}\left(\int_{\zeta^{-1}(x)}^{\zeta(x)}\left(\frac{\log\frac{\zeta(x)}{t}}{V(t)}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right)^{\frac{q(p-1)}{p-q}}dx\right)^{\frac{p-q}{pq}}.$$

Finally, we obtain the following.

Theorem 10. *Let $0 < p, q < \infty$. Then for the maximal Hardy-Littlewood operator*

$$\|M\|_{\Gamma^p(v)\to\Gamma^q(u)} \approx \mathscr{A}_0 + \mathscr{A}_2 + \mathbf{A}_0 + \mathbf{A}_2.$$

С ДНЕМ РОЖДЕНИЯ !

THANK YOU FOR ATTENTION !