Hirzebruch genera and functional equations

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International Conference
Geometry Days in Novosibirsk — 2014
dedicated to 85th anniversary of
Yuri Grigorievich Reshetnyak

September 24–27, 2014
Novosibirsk, Russia

We will consider a smooth oriented manifold
with a smooth action of a compact torus,
such that all fixed points are isolated.

Such manifolds naturally appear
in different areas of mathematics.

They are the key objects of toric geometry, toric topology,
and the theory of homogeneous spaces of compact Lie groups.

The theory of Hirzebruch genera of manifolds
is a well-known area of algebraic topology.
It has important applications
in the theory of differential operators on manifolds,
mathematical physics and combinatorics.

In the case of manifolds with compact torus action
there is an equivariant Hirzebruch genus
and arises the famous rigidity problem for this genus.

In many cases this problem is equivalent to the problem
of fiberwise multiplicativity of Hirzebruch genera.

The localization formulas for equivariant genus appear.
They give the value of this genus in terms of torus representation in the tangent space at fixed points.

The rigidity conditions and localization formulas lead to functional equations that characterize the fundamental fiberwise multiplicative genera.

In the talk we will describe the general approach to rigid Hirzebruch genera problem and demonstrate the results for the homogeneous manifolds of compact Lie groups.

The main construction

Let us consider

a set $\Lambda = \{\Lambda_i, i = 1, \ldots, m\}$ of $(k \times n)$-matrices Λ_i with integer coefficients

and a map $\varepsilon : [1, m] \to \{-1, 1\}$.

Let A be a commutative associative ring over $\mathbb{Q}$.

We associate to each series $f(x) = x + a_1 x^2 + a_2 x^3 + \ldots \in A[[x]]$ the characteristic function of the pair (Λ, ε):

$$L(\Lambda, \varepsilon; f)(t) = \sum_{i=1}^{m} \varepsilon(i) \prod_{j=1}^{n} \frac{1}{f(\langle \Lambda_i^j, t \rangle)}. \tag{1}$$

Here $t = (t_1, \ldots, t_k)$, $\Lambda_i^j, j = 1, \ldots, n$ are k-dimensional column vectors of Λ_i and $\langle \Lambda_i^j, t \rangle = \Lambda_i^{j,1} t_1 + \ldots + \Lambda_i^{j,k} t_k$.

Admissible pairs

Set

$$f(x) = \frac{x}{Q(x)}, \qquad Q(0) = 1.$$

We have:

$$L(\Lambda, \varepsilon; f)(t) = \sum_{i=1}^{n} \varepsilon(i) \left(\prod_{j=1}^{n} \frac{1}{\langle \Lambda_i^j, t \rangle} \right) \prod_{j=1}^{n} Q\left(\langle \Lambda_i^j, t \rangle \right). \tag{2}$$

The pair (Λ, ε) is called admissible if

$$L(\Lambda, \varepsilon; f)(t) \in A[[t]]$$

for any ring A and any series

$$f(x) = x + a_1 x^2 + a_2 x^3 + \ldots \in A[[x]].$$

The universal series

It is sufficient to check that the pair (Λ, ε) is admissible for the universal series

$$f_u(x) = x + \sum_{q \geqslant 1} a_q x^{q+1} \in \mathcal{A}[[x]],$$

where $\mathcal{A} = \sum_{n \geqslant 0} \mathcal{A}_{-2n} = \mathbb{Q}[a_1, \ldots, a_q, \ldots]$, $\deg a_q = -2q$.

Set $\deg t_l = 2$ for $l = 1 \ldots, k$. If the pair (Λ, ε) is admissible, then

$$L(\Lambda, \varepsilon; f_u)(t) = \sum_{\omega} P_\omega t^\omega, \tag{3}$$

where each $\omega = (i_1, \ldots, i_k)$ is a set of non-negative integers, $t^\omega = t_1^{i_1} \ldots t_k^{i_k}, \quad |\omega| = i_1 + \ldots + i_k$ and $P_\omega \in \mathcal{A}_{-2(n+|\omega|)}$.

Note $L(\Lambda, \varepsilon; f_u)(0) = P(a_1, \ldots, a_n)$, where $P(\cdot) = P_\varnothing(\cdot)$, $\deg P_\varnothing = -2n$.

Rigid pairs

The pair (Λ, ε) is called rigid for a family of series $\mathcal{F}$ if

$$L(\Lambda, \varepsilon; f)(t) \equiv L(\Lambda, \varepsilon; f)(0) = P(a_1, \ldots, a_n) \in A$$

for any series $f \in \mathcal{F}$.

Problem

Find the solution of rigidity functional equation

$$L(\Lambda, \varepsilon; f)(t) \equiv C$$

where C is constant in t,
that is, for a given pair (Λ, ε), find the family of series $\mathcal{F}$
and calculate the polynomial $C = P(a_1, \ldots, a_n)$.

Manifolds with torus action

Theorem

For any smooth oriented manifold M^{2n} with a smooth action of the compact torus T^k such that all the fixed points are isolated there is the correspondence

$$\mathcal{L} : (M^{2n}, T^k) \to (\Lambda, \varepsilon).$$

Proof. Let $x_1, \ldots, x_m$ be the set of all fixed points. Then in the tangent space $\tau_i \simeq \mathbb{R}^{2n}$ of the point x_i a representation of the torus T^k is defined.

Given a basis in T^k one can choose a set of weights

$$\Lambda_i^j = \{\Lambda_i^{j,1}, \ldots, \Lambda_i^{j,k}\}, \quad j = 1, \ldots, n.$$

Manifolds with torus action

One can define the map

$$\varepsilon : [1, m] \to \{-1, 1\},$$

where $\varepsilon(i) = 1$,
if the orientation in τ_i, induced by the orientation of the manifold M^{2n}, coincides with the orientation in τ_i, defined by the set of weights Λ_i^j,
and $\varepsilon(i) = -1$ otherwise.

Therefore we have the correspondence $\mathcal{L}$.

Normal complex T^k-manifolds

Let (M^{2n}, T^k) be a smooth manifold M^{2n} with an action of a torus T^k.

There is a linear representation of the torus T^k in $\mathbb{R}^{2N} \simeq \mathbb{C}^N$ and an equivariant embedding $M^{2n} \subset \mathbb{C}^N$.
Let $\nu_N(M^{2n})$ be the normal bundle of this embedding.

The pair (M^{2n}, T^k) is called normal complex T^k-manifold if there exists N such that $\nu_N(M^{2n})$ is a complex T^k-bundle.

If (M^{2n}, T^k) is a normal complex T^k-manifold, then M^{2n} is a stably-complex T^k-manifold and therefore it is orientable.

Hirzebruch genus (complex case)

Let

$$f(x) = x + \sum_{q \geqslant 1} a_q x^{q+1} \in A[[x]], \quad \text{as before.}$$

The series

$$\prod_{i=1}^{n} \frac{t_i}{f(t_i)}$$

can be presented in the form $L_f(\sigma_1, ..., \sigma_n)$, where σ_k is the k-th elementary symmetric polynomial of $t_1, ..., t_n$.

We have $\quad L_f(\sigma_1, ..., \sigma_n) = 1 - a_1\sigma_1 + (a_1^2 - a_2)\sigma_1^2 + (2a_2 - a_1^2)\sigma_2 + \dots$

The Hirzebruch genus L_f of a stably complex manifold M^{2n} with tangent Chern classes $c_i = c_i(\tau(M^{2n}))$ and fundamental cycle $\langle M^{2n} \rangle$ is defined by the formula

$$L_f(M^{2n}) = (L_f(c_1, ..., c_n), \langle M^{2n} \rangle) \in A_{-2n}.$$

The universal series $f_u(x)$ determines the isomorphism

$$L_{f_u} : \Omega_U \otimes \mathbb{Q} \to \mathbb{Q}[a_1, \ldots, a_q, \ldots],$$

where Ω_U is the ring of cobordisms of stably-complex manifolds and a_q, $q = 1, 2, \ldots$ are the coefficients of f.

Any series $f(x) \in A[[x]]$ gives a ring homomorphism

$$L_f : \Omega_U \to A.$$

Equivariant genus

Let (M^{2n}, T^k) be a normal complex T^k-manifold M^{2n} with an action of a torus T^k.
Then for any series $f(x)$ there is the equivariant genus

$$L_f(M^{2n}, T^k)(t) = L_f([M^{2n}]) + \sum_{|\omega|>0} Q_\omega t^\omega,$$

where $Q_\omega = L_f(B_\omega^{2(n+|\omega|)})$.
Here $[M^{2n}] \in \Omega_U^{-2n}$ is the complex cobordism class of M^{2n} and $B_\omega^{2(n+|\omega|)} \in \Omega_U^{-2(n+|\omega|)} \otimes \mathbb{Q}$ for all ω.

The construction of admissible pairs

From localization theorem for equivariant genus
(V. Buchstaber, T. Panov, N. Ray IMRN, 2010), we obtain

Corollary

Let (M^{2n}, T^k) be a normal complex T^k-manifold with isolated fixed points. Then the correspondence

$$\mathcal{L} : (M^{2n}, T^k) \to (\Lambda, \varepsilon)$$

gives the admissible pair (Λ, ε) and

$$L_f(M^{2n}, T^k)(t) = L(\mathcal{L}(M^{2n}, T^k), f)(t).$$

In particular, for every $\mathcal{L}(M^{2n}, T^k)$ the equation holds:

$$\sum_{i=1}^{m} \varepsilon(i) \prod_{j=1}^{n} \frac{1}{\langle \Lambda_i^j, t \rangle} \equiv 0.$$

Complex and almost complex manifolds

A pair (M^{2n}, T^k) is called a complex T^k-manifold,
if M^{2n} is a complex manifold
with a holomorphic action of a torus T^k.

A pair (M^{2n}, T^k) is called an almost complex T^k-manifold,
if on the tangent bundle $\tau(M^{2n})$
there exists a structure of a complex T^k-bundle.

The structure of
a complex or almost complex T^k-manifold (M^{2n}, T^k)
defines the structure of a normal complex manifold (M^{2n}, T^k)
and therefore an admissible pair (Λ, ε).

For each such pair $\varepsilon(i) = 1$, $i = 1, \dots, m$.

Complex projective spaces

$$\mathbb{C}P^n = \{(z_1 : \ldots : z_{n+1}); (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}\}$$

has the canonical structure of T^{n+1}-complex manifold with the fixed points $e_k = (\delta_k^1, \ldots, \delta_k^{n+1})$, $k = 1, \ldots, n+1$, $\delta_k^i = 0$ if $i \neq k$ and $\delta_k^k = 1$.

The weights at e_k are the n-dimensional vectors such that $\langle \Lambda_j^k, t \rangle = t_j - t_k$, $j \neq k$, and the signs are $\varepsilon(e_k) \equiv 1$.

For any series $f(x) \in A[[x]]$ such that $f(0) = 0$, $f'(0) = 1$ we get

$$\sum_{i=1}^{n+1} \prod_{j \neq i} \frac{1}{f(t_j - t_i)} \in A[[t_1, \ldots, t_{n+1}]].$$

Complex projective line

$$\mathbb{C}P^1 = \{(z_1 : z_2); (z_1, z_2) \in \mathbb{C}^2\}.$$

The action of T^2 on $\mathbb{C}P^1$: $(z_1 : z_2) \to (t_1 z_1 : t_2 z_2)$
has two fixed points $(1 : 0)$ and $(0 : 1)$.

Rigidity functional equation:

$$\frac{1}{f(t_2 - t_1)} + \frac{1}{f(t_1 - t_2)} \equiv C, \quad \text{where } f(x) = x + \dots, \quad C = -2a_1.$$

The general analytic solution of this equation is

$$f(x) = \frac{x}{q(x^2) - a_1 x}, \text{ where } q(0) = 1.$$

Hirzebruch L-genus — the signature of the manifold

Rigidity functional equation for $\mathbb{C}P^2$ is

$$\frac{1}{f(t_1-t_2)f(t_1-t_3)}+\frac{1}{f(t_2-t_1)f(t_2-t_3)}+\frac{1}{f(t_3-t_1)f(t_3-t_2)}\equiv C.$$

From this equation we get

$$C=3(2a_1^2-a_2), \qquad (2a_1^2-a_2)(a_1^3-2a_1a_2+a_3)^2=0.$$

If $f(x)$ is a solution of this equation and $f(-x)=-f(x)$, then

$$f(x+y)=\frac{f(x)+f(y)}{1+Cf(x)f(y)},$$

that is $f(x)=\frac{1}{\sqrt{C}}th(\sqrt{C}x)$.
This series determines the most famous Hirzebruch genus, namely, *the signature*.

Let the bundle $\mathbb{C}P(\xi) \to B$
with fiber $\mathbb{C}P(2)$ be the projectivization
of a 3-dimensional complex vector bundle $\xi \to B$.

A Hirzebruch genus $L_f : \Omega_U \to R$ is called $\mathbb{C}P(2)$-*multiplicative*,
if we have $L_f[\mathbb{C}P(\xi)] = L_f[\mathbb{C}P(2)]L_f[B]$.

If a genus L_f is $\mathbb{C}P(2)$-multiplicative, then it is rigid on $\mathbb{C}P(2)$.

Definition

We will call *special* $\mathbb{C}P(2)$-*multiplicative genus*
a $\mathbb{C}P(2)$-multiplicative genus L_f
such that $L_f[\mathbb{C}P(2)] = 0$.

Theorem (V. Buchstaber, E. Netay 2014)

Let L_f be a $\mathbb{C}P(2)$-multiplicative genus.
If $L_f[\mathbb{C}P(2)] \neq 0$, then L_f is the two-parametric Todd genus, and

$$f(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}}, \tag{4}$$

If $L_f[\mathbb{C}P(2)] = 0$, that is L_f is a special $\mathbb{C}P(2)$-multiplicative genus, then it is the two-parametric general elliptic genus and

$$f(x) = -\frac{2\wp(x) + \frac{a^2}{2}}{\wp'(x) - a\wp(x) + b - \frac{a^3}{4}}. \tag{5}$$

Here $\wp$ and $\wp'$ are Weierstrass functions of the elliptic curve with parameters $g_2 = -\frac{1}{4}(8b - 3a^3)a$, $g_3 = \frac{1}{24}(8b^2 - 12a^3b + 3a^6)$, discriminant $\Delta = -b^3(3b - a^3)$.

In terms of coefficients of $f(x)$ we have:

in the first case
$2a_1 = -(\alpha+\beta), \quad 3a_2 = \alpha\beta + 2a_1^2, \quad a_3 = 2a_1 a_2 - a_1^3,$

in the second case
$2a_1 = -a$, $2a_2 = a^2$, $8a_3 = 4b - 3a^3$.

Corollary

In the case $a_2 = 2a_1^2, \quad a_3 = 3a_1^3$, we have the "intersection case":

$$f(x) = \frac{2}{k}\frac{tg(y)}{tg(y)+\sqrt{3}}, \qquad y = \frac{\sqrt{3}}{2}kx. \tag{6}$$

Krichever genus

Consider the series

$$f(x) = \frac{e^{a_1 x}}{\Phi(x)}, \tag{7}$$

where

$$\Phi(x) = \Phi(x; g_2, g_3) = \frac{\sigma(x+\tau)}{\sigma(x)\sigma(\tau)} e^{-\zeta(\tau)x}$$

is Baker-Akhiezer function of the elliptic curve with Weierstrass parameters g_2, g_3.

Here $\sigma(x) = \sigma(x; g_2, g_3)$ and $\zeta(x) = \zeta(x; g_2, g_3)$ are Weierstrass functions.

Krichever genus

Definition

The Hirzebruch genus determined by the series (7)

$$f(x) = \frac{e^{a_1 x}}{\Phi(x)},$$

is called *Krichever genus*.

For this genus the following important result holds:

Theorem (I. Krichever, 1990)

The equivariant genus L_f determined by the series (7) is rigid on SU-manifolds with torus action.

The Baker-Akhiezer function $\Phi(x)$ can be decomposed in series whose coefficients are polynomials of $\wp(\tau), \wp'(\tau)$ and g_2.

Lemma

The equality (7) corresponds to the isomorphism

$$\mathbb{Q}[a_1, a_2, a_3, a_4] \to \mathbb{Q}[a_1, \wp(\tau), \wp'(\tau), g_2], \tag{8}$$

where

$$a_1 \mapsto a_1, \quad a_2 \mapsto \frac{1}{2}(\wp(\tau)+a_1^2), \quad a_3 \mapsto \frac{1}{6}\left(\wp'(\tau)+3a_1\wp(\tau)+a_1^3\right),$$
$$a_4 \mapsto \frac{1}{24}\left(9\wp(\tau)^2 - \frac{3}{5}g_2 + 4a_1\wp'(\tau) + 6a_1^2\wp(\tau) + a_1^4\right).$$

Corollary

The coefficients a_k, $k > 4$ of the series $f(x)$ for Krichever genus can be expressed using (8) as polynomials in a_1, a_2, a_3, a_4.

Theorem (V. Buchstaber, E. Netay 2014)

The special $\mathbb{C}P(2)$-multiplicative genus is a special Krichever genus, and (5) can be written in the form

$$f(x) = \frac{\sigma(x)\sigma(\tau)}{\sigma(x+\tau)} \exp\left(-\frac{a}{2}x + \zeta(\tau)x\right)$$

for the Baker-Akhiezer function with parameters

$$g_2 = \frac{3}{4}(24b + a^3)a, \quad g_3 = -\frac{1}{8}(72b^2 + 60a^3b - a^6),$$
$$\Delta = -81b(3b - a^3)^3.$$

The parameter τ is determined by the relations

$$\wp(\tau, g_2, g_3) = \frac{3}{4}a^2, \qquad \wp'(\tau, g_2, g_3) = 3b - a^3.$$

Corollary

Each $\mathbb{C}P(2)$-multiplicative genus is rigid on manifolds with S^1-equivariant SU-structure.

The parameters of the special $\mathbb{C}P(2)$-multiplicative genus form a manifold

$$\mathcal{K} = \left\{(a_1, a_2, a_3, a_4) : a_2 = 2a_1^2, \quad 5a_4 = -2a_1(8a_1^3 - 7a_3)\right\}$$

in the space of parameters of Krichever genus.

Let us note that expression of the series $f(x)$ for the special $\mathbb{C}P(2)$-multiplicative genus in terms of Weierstrass $\wp$-function and Baker-Akhiezer function correspond to different functions g_2 and g_3 in $\mathcal{K}$:

in the first case

$$g_2 = 4a_1(2a_3 - 3a_1^3), \quad g_3 = \frac{4}{3}a_3^2 - 4a_1^6,$$

in the second case

$$g_2 = -12a_1(6a_3 - 19a_1^3), \quad g_3 = -4(9a_3^2 - 84a_1^3a_3 + 169a_1^6).$$

Homogeneous spaces of compact Lie groups

Let G be a compact connected Lie group,
H its connected compact subgroup having the same rank as G
and T^k their common maximal torus.

On a smooth oriented manifold $M^{2n} = G/H$
the left action of the group G is defined.
This action induces a smooth action of T^k
with isolated fixed points $x_1, \ldots, x_m$,
where x_1 is the image of identity $e \in G$
under the projection $G \to M^{2n}$,
$x_i = w_i x_1$, where w_i are elements of the Weyl group $W(G)$
and $m = |W(G)/W(H)|$.

Homogeneous spaces of compact Lie groups

In the correspondence

$$\mathcal{L} : (M^{2n}, T^k) \to (\Lambda, \varepsilon)$$

we get

$$\Lambda_i^j = w_i \Lambda_1^j,$$

where $\{\Lambda_1^j\}$ is a set weights of the representation the torus T^k in the quotient of the Lie algebra $\mathcal{G}(G)$ by Lie subalgebra $\mathcal{G}(H)$.

Homogeneous spaces of compact Lie groups

In the case $H_2(M^{2n};\mathbb{Z}) \neq 0$
a homogeneous space $M^{2n} = G/H$ is a complex T^k-manifold.

Examples:

- Complex flag manifolds:

$$F_n = U(n)/T^n.$$

- Complex Grassmann manifolds:

$$G_{n,q} = U(n)/(U(q) \times U(n-q)).$$

- Generalized complex flag manifolds:

$$G_{n,q_1,\ldots,q_l} = U(n)/(U(q_1) \times \ldots \times U(q_l)),$$

where $q_1 + \ldots + q_l = n.$

Homogeneous spaces of compact Lie groups

In the case $H_2(M^{2n}; \mathbb{Z}) = 0$ a homogeneous space $M^{2n} = G/H$, where H is the centralizer of some element $g \in G$ of odd order, has a G-invariant almost complex structure.

Example.
The sphere $S^6 = G_2/SU(3)$ has a G_2-invariant almost complex structure, because $SU(3)$ is the centralizer of an element $g \in G_2$ of order 3, which generates the center of G_2.

This almost complex structure on S^6 is not integrable.

Almost complex T^2-manifold S^6

The action of $T^2 \subset G_2$ on S^6
has two fixed points x_1 and x_2 with weights

$$\begin{array}{llll} x_1: & \Lambda_1^1 = (1,0), & \Lambda_1^2 = (0,1), & \Lambda_1^3 = (-1,-1), \\ x_2: & \Lambda_2^1 = (-1,0), & \Lambda_2^2 = (0,-1), & \Lambda_2^3 = (1,1). \end{array}$$

Rigidity functional equation:

$$\frac{1}{f(t_1)f(t_2)f(-t_1-t_2)} + \frac{1}{f(-t_1)f(-t_2)f(t_1+t_2)} \equiv C.$$

Almost complex T^2-manifold S^6

Set $t_1 = x$, $t_2 = y$. The rigidity equation becomes

$$\frac{1}{f(x)f(y)f(-x-y)} + \frac{1}{f(-x)f(-y)f(x+y)} = C. \tag{9}$$

Set

$$b(x) = -\frac{f(x)}{f(-x)} = 1 + \sum_{k\geqslant 1} b_k x^k.$$

We get the equation

$$b(x+y) = b(x)b(y) - Cf(x)f(y)f(x+y). \tag{10}$$

For $C = 0$ we obtain $b(x) = e^{-\mu x}$
and the function $f(x)$ is characterized by the relation

$$f(-x) = -e^{\mu x} f(x).$$

Almost complex T^2-manifold S^6

Let $C \neq 0$. Using the operator

$$\partial = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

from (10) we get

$$Cf(x+y) = \frac{b'(x)b(y) - b(x)b'(y)}{f'(x)f(y) - f(x)f'(y)}.$$

For $y = 0$ this equation becomes

$$b'(x) = b_1 b(x) - Cf(x)^2,$$

therefore we get $C = \frac{1}{2}(b_1^3 - b_3)$.
From these relations we get

$$f(x+y) = \frac{f(x)^2 b(y) - b(x)f(y)^2}{f(x)f'(y) - f'(x)f(y)}. \tag{11}$$

Almost complex T^2-manifold S^6

From a theorem by V. Buchstaber, 1990, we obtain

Corollary

The general analytic solution of equation (11) *is given by the function*

$$f(x) = \frac{e^{\lambda x}}{\Phi(x)},$$

where

$$\Phi(x) = \frac{\sigma(\alpha - x)}{\sigma(x)\sigma(\alpha)} e^{\zeta(\alpha)x}$$

is the Baker-Akhiezer function.

Hirzebruch genus (oriented case)

Let

$$f(x) = x + \sum_{k\geqslant 1} a_{2k}x^{2k+1} \in A[[x]].$$

The product of even series

$$\prod_{i=1}^{n} \frac{t_i}{f(t_i)}$$

can be presented in the form $L_f(p_1, ..., p_n)$, where p_k is the k-th elementary symmetric polynomial in $t_1^2, ..., t_n^2$.

We have $\quad L_f(p_1, ..., p_n) = 1 - a_2 p_1 + (a_2^2 - a_4)p_1^2 + (2a_4 - a_2^2)p_2 + \dots$

The Hirzebruch genus L_f of a oriented manifold M^{4n} with tangent Pontriagin classes

$$p_k(\tau(M^{4n})) = (-1)^k c_{2k}(\tau_{\mathbb{C}}(M^{4n}))$$

and fundamental cycle $\langle M^{4n}\rangle$ is defined by the formula

$$L_f(M^{4n}) = (L_f(p_1, ..., p_n), \langle M^{4n}\rangle) \in A_{-4n}.$$

The construction of admissible pairs

From localization theorem for equivariant genus we obtain

Corollary

Let (M^{4n}, T^k) *be an oriented* T^k*–manifold* M^{4n}*.*
Then the correspondence

$$\mathcal{L} : (M^{4n}, T^k) \to (\Lambda, \varepsilon)$$

gives the admissible pair (Λ, ε) *and*

$$\mathcal{L}_f(M^{4n}, T^k) = L(\mathcal{L}(M^{4n}, T^k), f).$$

Let (M^{4n}, T^k) be an oriented T^k-manifold.
For any odd series $f(x) = x + \sum_{q\geqslant 1} a_{2q} t^{2q+1}$
is defined the equivariant genus

$$L_f(M^{4n}, T^k) = L_f[M^{4n}] + \sum Q_\omega t^\omega$$

where $Q_\omega = L_f(c_\omega^{4(n+|\omega|)})$.
Here $[M^{4n}] \in \Omega_{SO}^{-4n}$ is oriented cobordism class of M^{4n}
and $c_\omega^{4(n+|\omega|)} \in \Omega_{SO}^{-4(n+|\omega|)} \otimes \mathbb{Q}$ for all ω.
For the ring of oriented cobordisms Ω_{SO} the epimorphism holds:

$$\mu_U^{SO} : \mathbb{Q}[a_1, \dots, a_q, \dots] = \Omega_U \otimes \mathbb{Q} \to \Omega_{SO} \otimes Q = Q[a_2, \dots, a_{2q}, \dots],$$

where $\mu_U^{SO}(a_{2q}) = a_{2q}$, $\mu_U^{SO}(a_{2q+1}) = 0$.

Quaternionic projective spaces

$$\mathbb{H}P^n = \{(q_1 : \ldots : q_{n+1}); (q_1, \ldots, q_{n+1}) \in \mathbb{H}^{n+1}\}$$
$$\text{where} \quad (q_1, \ldots, q_{n+1}) = (q_1 q, \ldots, q_{n+1} q), \quad q \in \mathbb{H}\backslash 0.$$

The oriented manifold $\mathbb{H}P^n$
has the canonical structure of T^{n+1}-manifold
with fixed points $e_k = (\delta_k^1, \ldots, \delta_k^{n+1})$, $k = 1, \ldots, n+1$, where

$$(t_1, \ldots, t_{n+1})(q_1, \ldots, q_{n+1}) = (t_1 q_1, \ldots, t_{n+1} q_{n+1}).$$

The weights at e_k are the $2n$-dimensional vectors
$\{(\Lambda_k^{j,+}, \Lambda_k^{j,-}), j \neq k\}$ such that $\langle \Lambda_k^{j,\pm}, t \rangle = t_j \pm t_k$.

For any odd series $f(x) \in A[[x]]$ we get

$$\sum_{k=1}^{n+1} \prod_{j \neq k} \frac{1}{f(t_j + t_k) f(t_j - t_k)} \in A[[t_1^2, \ldots, t_{n+1}^2]].$$

Quaternionic projective line

$$\mathbb{H}P^1 = \{(q_1 : q_2); (q_1, q_2) \in \mathbb{H}^2\}.$$

The action of T^2 on $\mathbb{H}P^1$

$$(q_1 : q_2) \to (t_1 q_1 : t_2 q_2)$$

has fixed points $e_1 = (1 : 0)$ and $e_2 = (0 : 1)$. The equivariant genus is

$$L_f(\mathbb{H}P^1, T^2) = \frac{1}{f(t_2 + t_1) f(t_2 - t_1)} + \frac{1}{f(t_1 + t_2) f(t_1 - t_2)}.$$

In oriented cobordisms $[\mathbb{H}P^1] = 0$ the condition

$$L_f(\mathbb{H}P^1, T^2)(0) = L_f[\mathbb{H}P^1] = 0$$

is provided by the condition that $f(x)$ is odd.

Rigidity equation for $\mathbb{H}P^2$

For any odd series $f(x) = x + \dots$ by setting $t_1 = x$, $t_2 = y$, $t_3 = z$ we have

$$\frac{1}{f(y+x)f(y-x)f(z+x)f(z-x)} + \\ + \frac{1}{f(x+y)f(x-y)f(z+y)f(z-y)} + \\ + \frac{1}{f(x+z)f(x-z)f(y+z)f(y-z)} = C. \quad (12)$$

By setting $z = 0$ and using that $f(x)$ is odd we get the functional equation

$$f(x+y)f(x-y) = \frac{f(x)^2 - f(y)^2}{1 - Cf(x)^2 f(y)^2}. \tag{13}$$

Theorem

The general analytic solution of (13) *is the function satisfying the differential equation*

$$f'(x)^2 = 1 + 3a_2 f(x)^2 - Cf(x)^4$$

with initial conditions $f(0) = 0$, $f'(0) = 1$, that is $f(x) = sn(x)$ is elliptic Jacobi sine.

Proof.

Decomposing the left and right hand side of (13) as a series in y and equating the coefficients at y^2 we get the equation

$$(f')^2 = 1 + ff'' - Cf^4. \tag{14}$$

Using that $f'(x)$ is even and $f'(0) = 1$ we can set

$$(f')^2 = 1 + \sum b_k f^{2k}.$$

Hence $ff'' = \sum kb_k f^{2k}$. Now from (14) we immediately obtain $b_1 = 3a_2$, $b_2 = C$ and $b_k = 0$ for $k > 2$.
Therefore if $f(x)$ satisfies (12),
then it's necessary that $f(x) = sn(x)$.
From the classical addition theorem for Jacobi elliptic sine it follows that $sn(x)$ satisfies (12).

Ochanine genus

Definition

The Hirzebruch genus determined by the series $f(x) = sn(x)$ is called *Ochanine genus*.

Theorem (Ochanine, Bott – Taubes)

Ochanine genus is fiberwise multiplicative for bundles $E \to B$ *of oriented manifolds with fiber* M *being a spin-manifold, that is* $w_2(M) \equiv 0$ *where* w_2 *is the second Stiefel–Whitney class in ordinary cohomology.*

Let B be an oriented manifold
and $\mathbb{H}P(\xi) \to B$ be a bundle with fiber $\mathbb{H}P(2)$,
which is a quaternization of the vector bundle $\xi \to B$.

The Hirzebruch genus $L_f : \Omega_{SO} \to R$
is called $\mathbb{H}P(2)$-multiplicative if

$$L_f[\mathbb{H}P(\xi)] = L_f[\mathbb{H}P(2)]L_f[B].$$

From a theorem of V. Buchstaber, T. Panov, N. Ray we obtain:

Corollary

Each $\mathbb{H}P(2)$-multiplicative genus is rigid on $\mathbb{H}P(2)$.

Theorem

The Hirzebruch genus L_f is fiberwise multiplicative for bundles of oriented manifolds whose fibers are spin-manifolds if and only if it is Ochanine genus.

Addendum.

Flag manifolds $U(n)/T^n$

Universal rigidity rings for flag manifolds

$$F_n = U(n)/T^n$$

Using of divided difference operators

Addendum: Flag manifolds $U(n)/T^n$.

We consider $U(n)$-invariant complex structure on $U(n)/T^n$.

Recall that the Weyl group $W_{U(n)}$ is the symmetric group S_n
and it permutes the coordinates $x_1, \ldots, x_n$
on Lie algebra $\mathfrak{t}^n$ for T^n.

The canonical action of the torus T^n on this manifold has

$$\|W_{U(n)}\| = \chi(U(n)/T^n) = n!$$

fixed points and its weights at identity point
are given by the roots of $U(n)$.

Addendum: Universal rigidity rings for flag manifolds $F_n = U(n)/T^n$.

Let

$$\Delta_n = \prod_{1\leqslant i<j\leqslant n} (x_i - x_j).$$

Theorem

The rigidity functional equation for flag manifolds F_n is

$$C\Delta_n = \sum_{\sigma\in S_n} (\mathrm{sign}\sigma)\sigma \prod_{1\leqslant i<j\leqslant n} Q(x_i - x_j), \tag{15}$$

where $Q(t) = 1 + \sum\limits_{i\geqslant 1} b_i t^i$ and C is a homogeneous degree $-2n$ polynomial in $b_1, \ldots, b_n$, $\deg b_k = -2k$.

Here $\mathrm{sign}\sigma$ is the sign of the permutation σ.

Addendum: $F_3 = U(3)/T^3$

$$\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

$$C = 6(b_1^3 + b_1 b_2 - b_3).$$

$$C\Delta_3 = \sum_{\sigma \in S_3} (\mathrm{sign}\sigma)\sigma\Big(Q(x_1 - x_2)Q(x_1 - x_3)Q(x_2 - x_3)\Big).$$

The first generators of the rigidity ideal are

$$5b_5 = b_1 b_2^2 + 6b_1^2 b_3 - b_2 b_3 + 5b_1 b_4,$$
$$7b_7 = 3b_1 b_3^2 + 2b_1 b_2 b_4 + b_3 b_4 + 6b_1^2 b_5 - 3b_2 b_5 + 7b_1 b_6.$$

Addendum: Using of divided difference operators.

Consider the ring of the symmetric polynomials

$$\mathsf{Sym}_n \subset \mathbb{Z}[x_1, \ldots, x_n].$$

There is a linear operator

$$L : \mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \mathsf{Sym}_n : \ L\mathbf{x}^\xi = \frac{1}{\Delta_n} \sum_{\sigma \in S_n} (\mathrm{sign}\sigma)\sigma\mathbf{x}^\xi \ ,$$

where $\xi = (j_1, \ldots, j_n)$ and $\mathbf{x}^\xi = x_1^{j_1} \cdots x_n^{j_n}$.

It follows from the definition of Schur polynomials

$$\mathsf{Sh}_\lambda(x_1, \ldots, x_n), \quad \text{where} \quad \lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$$

that

$$L\mathbf{x}^{\lambda+\delta} = \mathsf{Sh}_\lambda(x_1, \ldots, x_n),$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$ and $L\mathbf{x}^\delta = 1$.

Here

$$\mathbf{x}^\delta = x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

For $n = 3$

$$x^\delta = x_1^2 x_2.$$

Moreover, the operator L have the following properties:

- $L\mathbf{x}^{\xi} = 0$, if $j_1 \geqslant j_2 \geqslant \cdots \geqslant j_n \geqslant 0$ and $\xi \neq \lambda + \delta$ for some $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$;
- Let $\xi = (j_1, \ldots, j_n)$ and $\sigma \in S_n$ such that $\sigma\xi = \xi'$, where $\xi' = (j_1', \ldots, j_n')$, $j_1' \geqslant \ldots \geqslant j_n'$, then
$$L\mathbf{x}^{\xi} = (\mathrm{sign}\sigma) L\mathbf{x}^{\xi'};$$
- L is a homomorphism of Sym_n-modules.

We have
$$\prod_{1 \leqslant i < j \leqslant n} Q(x_i - x_j) = 1 + \sum_{|\xi| > 0} P_{\xi}(\mathbf{b})\mathbf{x}^{\xi}.$$

Here $|\xi| = \xi_1 + \ldots + \xi_n$, and $\mathbf{b} = (b_1, \ldots, b_k, \ldots)$.

Let us introduce the action of S_n on polynomials $P_\xi(\mathbf{b})$ by

$$1 + \sum_{|\xi|>0} (\sigma P_\xi(\mathbf{b}))\mathbf{x}^\xi = \sigma^{-1} \prod_{1\leqslant i<j\leqslant n} Q(x_i - x_j),$$

where $\sigma \in S_n$ on the right acts
by the permutation of variables $x_1, \ldots, x_n$.

Directly from the definition we have

$$1 + \sum_{|\xi|>0} (\sigma P_\xi(\mathbf{b}))\mathbf{x}^\xi = 1 + \sum_{|\xi|>0} P_\xi(\mathbf{b})(\sigma^{-1}\mathbf{x}^\xi).$$

Therefore $$\sigma P_\xi = P_{\sigma\xi}.$$

Corollary

- *The operator*
$$L^* = \sum_{\sigma \in S_n} (\mathrm{sign}\sigma)\sigma$$
acts on polynomials $P_\xi(\mathbf{b})$.
- *The formula holds*
$$C = \sum_{|\lambda| \geqslant 0} L^* P_{\lambda+\delta}(\mathbf{b})\, \mathrm{Sh}_\lambda(\mathbf{x}),$$
where
$\delta = (n-1, n-2, \ldots, 1, 0),\ \lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0)$.

Using that the Schur polynomials $\mathrm{Sh}_\lambda(\mathbf{x})$
form an additive basis in the ring of symmetric polynomials
we obtain the following result.

Using the action of S_n on polynomials $P_\xi(\cdot)$

Theorem

For standard structure on complex flag manifolds F_n and the canonical admissible pair (Λ, ε) we have

- $\varepsilon \equiv 1$;
- $C = L^* P_\delta(\mathbf{b})$;
- *Generators of rigidity ideal are* $L^* P_{\lambda+\delta}(\mathbf{b})$ *for all* $|\lambda| > 0$.

Remark

Polynomials $P_{\sigma\delta}$ in the formula for C appear to be polynomials only in variables $b_1, \ldots, b_{2n-3}$.

$F_3 = U(3)/T^3$

The generator

$$b_1 b_2^2 + 6 b_1^2 b_3 - b_2 b_3 + 5 b_1 b_4 - 5 b_5$$

of rigidity ideal for $F_3 = U(3)/T^3$
is the coefficient at

$$2\left(\mathrm{Sh}_{(2,0,0)} - 2\,\mathrm{Sh}_{(1,1,0)}\right).$$

Addendum.

Simple polytopes.

Moment-angle manifolds $\mathcal{Z}_P$.

Quasitoric manifolds $M(P, \Lambda_P)$.

Admissible pairs for quasitoric manifolds.

2-truncated cubes.

Addendum: Simple polytopes

A convex n-polytope $P \subset \mathbb{R}^n$ is called simple
if in every vertex exactly n facets converge.

Let $$P = \{x \in \mathbb{R}^n : \langle a_i, x_i \rangle + b_i \geqslant 0, 1 \leqslant i \leqslant m\}.$$

It is assumed that none of the inequalities can be removed.

Let us form a $(n \times m)$-matrix A_P,
whose columns are the vectors a_i in the standard basis.
We identify the polytope P with the intersection
of the n-dimensional plane

$$\{y \in \mathbb{R}^m : y = A_P^{\star} x + b\}$$

and the positive cone in $\mathbb{R}^m$.

Here and below $\star$ is the symbol of transposition.

Addendum: Moment-angle manifolds $\mathcal{Z}_P$

The manifold $\mathcal{Z}_P$ with the canonical action of the torus T^m is defined by the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_P \arrow[r] \arrow[d] & \mathbb{C}^m \arrow[d, "\rho"] \\ P \arrow[r] & \mathbb{R}^m_{\geqslant} \end{tikzcd}$$

It is called the moment-angle manifold.

Here $\rho : \mathbb{C}^m \to \mathbb{R}^m_{\geqslant} : \rho(z) = (|z_1|^2, \ldots, |z_m|^2)$.

Addendum: Quasitoric manifolds $M(P, \Lambda_P)$

Let $F_1, \ldots, F_m$ be the set of facets of a simple polytope P.
The $(n \times m)$-matrix Λ_P with integer coefficients
defines the characteristic mapping

$$\lambda : \{F_1, \ldots, F_m\} \to \mathbb{Z}^n; \quad \lambda(F_j) = \lambda_j$$

if for any vertex $v = F_{j_1} \cap \cdots \cap F_{j_n}$
the columns $\lambda_{j_1}, \ldots, \lambda_{j_n}$ form a basis in $\mathbb{Z}^n$.
The matrix Λ_P defines an epimorphism $\lambda : \mathbb{T}^m \to \mathbb{T}^n$.
The group $K(\Lambda_P) = \ker \lambda$ of rank $(m-n)$ acts freely on $\mathcal{Z}_\mathcal{P}$.
The orbit space $M^{2n} = \mathbb{Z}^n / K(\Lambda_P)$
is a smooth manifold called quasitoric.
An n-dimensional torus $T^n = \mathbb{T}^m / K(\Lambda_P)$ acts on M^{2n} with isolated fixed points, which are numbered by vertices of the polytope P.

Addendum: Admissible pairs for quasitoric manifolds

Each quasitoric manifold $M(P, \Lambda_P)$
is a normal complex T^n-manifold, where $n = \dim P$.

Let $v = F_{j_1} \cap \cdots \cap F_{j_n}$ be a vertex.
For a $(n \times n)$ matrix λ_v with columns $\{\lambda_{j_q}, q = 1, \ldots, n\}$
one can define a matrix with integer coefficients

$$\Lambda_v = (\lambda_v^\star)^{-1}.$$

Each quasitoric manifold corresponds to an admissible pair
(Λ, ε), where $\Lambda = \{\Lambda_v\}$, and

$$\varepsilon(v) = sign(\det(\lambda_{j_1}, \ldots, \lambda_{j_n}) \det(a_{j_1}, \ldots, a_{j_n})).$$

Addendum: 2-truncated cubes

A simple polytope is called 2-truncated cube if it is obtained by a sequence of truncations of the cube facets of codimension 2. The sequence of facets truncations, giving 2-truncated cube, is called its framing.

Let P be a 2-truncated cube with the $(n \times m)$-matrix Λ_P.
The truncation of the facet $G_{j_1,j_2} = F_{j_1} \cap F_{j_2} \neq \varnothing$
gives the polytope Q with $(n \times (m+1))$-matrix Λ_Q,
which is obtained from Λ_P
by adding a column $\lambda_{j_1} + \lambda_{j_2}$ on the $(m+1)$-th place.

The cube I^n has a canonical $(n \times 2n)$-matrix $\Lambda_{I^n} = (E_n, -E_n)$,
where E_n is the identity $(n \times n)$-matrix.
Each 2-truncated cube corresponds to a canonical matrix Λ_P,
which is obtained from Λ_{I^n} by the operations described above.

Addendum: 2-truncated cubes

One of the central results of the theory of 2-truncated cubes is the proof that flag nestohedra, graph-associahedra, graph-cubahedra, and other polytopes important in various fields of research are 2-truncated cubes.

Thus, we conclude that each of these classes of polyhedra has a canonical matrix Λ_P, and therefore, for each such polytope we obtain an admissible pair (Λ, ε), where $\varepsilon(i) = 1$ for all i.