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FUNK–MINKOWSKI TYPE TRANSFORMS OF VECTOR FIELDS ON THE SPHERE $\mathbb{S}^2$

SERGEY KAZANTSEV

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| = 1\}$, where $|\cdot|$ denotes the Euclidean norm. Throughout the paper we adopt the convention to denote in bold type the vectors in $\mathbb{R}^3$, and in simple type the scalars in $\mathbb{R}$. By the greek letters $\boldsymbol{\theta}$, $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ and so on we denote the units vectors on the sphere $\mathbb{S}^2$. The Funk–Minkowski transform $\mathcal{F}$ associates a function u or vector field $\mathbf{f}$ on the sphere $\mathbb{S}^2$ with its mean values (integrals) along all great circles of the sphere,

$$(1)\qquad \{\mathcal{F}\ \begin{matrix} u \\ \mathbf{f} \end{matrix}\ \}(\boldsymbol{\eta}) \equiv \mathcal{F}_{\boldsymbol{\eta}}\ \begin{matrix} u \\ \mathbf{f} \end{matrix}\ = \frac{1}{2\pi}\int_{\mathbb{S}^2} \begin{matrix} u(\boldsymbol{\theta}) \\ \mathbf{f}(\boldsymbol{\theta}) \end{matrix}\ \delta(\boldsymbol{\eta}\boldsymbol{\cdot}\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta},$$

where δ is the Dirac delta function and the $\mathrm{d}\boldsymbol{\theta}$ is the surface measure on $\mathbb{S}^2$ with normalization $\int_{\mathbb{S}^2} \mathrm{d}\boldsymbol{\theta} = 4\pi$. In the second case the Funk–Minkowski transform $\mathcal{F}$ is applied to vector function $\mathbf{f}$ by componentwise.

The spherical convolution operator $\mathcal{S}$ of Hilbert type is defined by,

$$\{\mathcal{S}\ \begin{matrix} u \\ \mathbf{f} \end{matrix}\ \}(\boldsymbol{\theta}) \equiv \mathcal{S}_{\boldsymbol{\theta}}\ \begin{matrix} u \\ \mathbf{f} \end{matrix}\ = \frac{\text{p.v.}}{4\pi}\int_{\mathbb{S}^2} \begin{matrix} u(\boldsymbol{\eta}) \\ \mathbf{f}(\boldsymbol{\eta}) \end{matrix}\ \frac{\mathrm{d}\boldsymbol{\eta}}{\boldsymbol{\theta}\boldsymbol{\cdot}\boldsymbol{\eta}},\ \ \boldsymbol{\theta} \in \mathbb{S}^2.$$

In addition, we also consider the following Funk-Minkowski type transforms of vector fields on the the sphere

$$(2)\qquad \{\mathcal{F}^{(\boldsymbol{\tau})}\mathbf{f}\}(\boldsymbol{\eta}) \equiv \mathcal{F}^{(\boldsymbol{\tau})}_{\boldsymbol{\eta}}\mathbf{f} = \frac{\boldsymbol{\eta}\boldsymbol{\cdot}}{2\pi}\int_{\mathbb{S}^2} \boldsymbol{\theta}\times\mathbf{f}(\boldsymbol{\theta})\delta(\boldsymbol{\eta}\boldsymbol{\cdot}\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta},$$

$$(3)\qquad \{\mathcal{F}^{(\boldsymbol{\beta})}\mathbf{f}\}(\boldsymbol{\eta}) \equiv \mathcal{F}^{(\boldsymbol{\beta})}_{\boldsymbol{\eta}}\mathbf{f} = \frac{\boldsymbol{\eta}\boldsymbol{\cdot}}{2\pi}\int_{\mathbb{S}^2} \mathbf{f}(\boldsymbol{\theta})\delta(\boldsymbol{\eta}\boldsymbol{\cdot}\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}.$$

The transform (2) will be an analog of the longitudinal ray transform of vector fields in the Euclidean case. In the physical sense, the quantity $\mathcal{F}^{(\boldsymbol{\tau})}_{\boldsymbol{\eta}}\mathbf{f}$ is equal to the circulation (work) of vector field $\mathbf{f}$ along the closed contour (big circle) $\boldsymbol{\theta}\boldsymbol{\cdot}\boldsymbol{\eta} = 0$ on the sphere.

The tangential gradient or the surface gradient, denoted by $\nabla \equiv \nabla_{\boldsymbol{\xi}}$ and the tangential rotated gradient (the surface curl-gradient), denoted by $\nabla^{\perp} \equiv \nabla^{\perp}_{\boldsymbol{\xi}}$, are defined accordingly as

$$(4)\qquad \nabla_{\boldsymbol{\xi}} u = \frac{\partial u}{\partial\theta}\mathbf{e}_1(\boldsymbol{\xi}) + \frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi}\mathbf{e}_2(\boldsymbol{\xi}), \quad \nabla^{\perp}_{\boldsymbol{\xi}} u = \boldsymbol{\xi}\times\nabla_{\boldsymbol{\xi}} u,$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the orthonormal basis in the tangent plane $\boldsymbol{\xi}^{\perp} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x}\boldsymbol{\cdot}\boldsymbol{\xi} = 0\}$,

$$\mathbf{e}_1(\boldsymbol{\xi}) = \frac{\partial\boldsymbol{\xi}}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta),\ \mathbf{e}_2(\boldsymbol{\xi}) = \frac{1}{\sin\theta}\frac{\partial\boldsymbol{\xi}}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0),$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\theta,\varphi) = \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi + \mathbf{k}\cos\theta = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

The surface divergence $\mathrm{div}_{\boldsymbol{\xi}}$ of vector-valued function $\mathbf{v}(\boldsymbol{\xi}) = v^1\mathbf{e}_1(\boldsymbol{\xi}) + v^2\mathbf{e}_2(\boldsymbol{\xi}) + v^3\boldsymbol{\xi}$ on the sphere $\mathbb{S}^2$ is written as,

$$(5)\qquad \mathrm{div}_{\boldsymbol{\xi}}\mathbf{v} = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(v^1\sin\theta) + \frac{\partial}{\partial\varphi}v^2\right) + 2v^3\,.$$

Finally, we define the Laplace–Beltrami operator $\Delta \equiv \Delta_{\boldsymbol{\xi}}$ as $\Delta_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = \mathrm{div}_{\boldsymbol{\xi}}\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi})\,.$

Theorem 1. *For any function $f(\boldsymbol{\theta}) \in H^1(\mathbb{S}^2)$ the following identity takes place*

(6)

$$f(\boldsymbol{\theta}) = \underbrace{\frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}}_{=f_{00}} + \frac{\text{p.v.}}{4\pi}\int_{\mathbb{S}^2}\frac{(\boldsymbol{\eta}+\boldsymbol{\theta})\,\boldsymbol{.}\,\Big\{\Big[\mathcal{F},\nabla\Big]f\Big\}(\boldsymbol{\eta})}{\boldsymbol{\eta}\,\boldsymbol{.}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} = f_{00} + \mathcal{S}_{\boldsymbol{\theta}}(\boldsymbol{\eta}+\boldsymbol{\theta})\,\boldsymbol{.}\,\Big[\mathcal{F},\nabla\Big]_{\boldsymbol{\eta}} f.$$

Here operators $\mathcal{F}$ and ∇ are the Funk–Minkowski transform (1) and the surface gradient (4), respectively. Through the square brackets $[.,.]$ we, as usual, denoted the commutator $\Big[\mathcal{F},\nabla\Big]f = \mathcal{F}\nabla f - \nabla\mathcal{F}f$.

We see that by using formula (6) the unknown function f completely reconstruct, if two Funk–Minkowski transforms, $\mathcal{F}f$ and $\mathcal{F}\nabla f$, are known.

Another result of this article is related to the problem of Helmholtz–Hodge decomposition for tangent vector field on the sphere $\mathbb{S}^2$. The Helmholtz–Hodge decomposition says that we can write any vector field tangent to the surface of the sphere as the sum of a curl-free component and a divergence-free component

$$\mathbf{f}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} u(\boldsymbol{\theta}) + \nabla^{\perp}_{\boldsymbol{\theta}} v(\boldsymbol{\theta}). \tag{7}$$

Here $\nabla_{\boldsymbol{\theta}} u$ is called also as inrrotational, poloidal, electric or potential field and $\nabla^{\perp}_{\boldsymbol{\theta}} v$ is called as incompressible, toroidal, magnetic or stream vector field. Scalar functions u and v are called velocity potential and stream functions, respectively.

In the next theorem we show that decomposition (7) is obtained by use of Funk–Minkowski-transform $\mathcal{F}$ and spherical convolution transform $\mathcal{S}$.

Theorem 2. *Any vector field $\mathbf{f} \in \mathbf{L}_{2,tan}(\mathbb{S}^2)$ that is tangent to the sphere can be uniquely decomposed into a sum (7) of a surface curl-free component and a surface divergence-free component with scalar valued functions $u, v \in H^1(\mathbb{S}^2)/\mathbb{C}$. Functions u and v are velocity potential and stream functions that are calculated unique up to a constant by the formulas*

$$u(\boldsymbol{\theta}) = \Big[\mathcal{S},\boldsymbol{\eta}\,\boldsymbol{.}\,,\mathcal{F}\Big]_{\boldsymbol{\theta}}\mathbf{f} = \Big\{\mathcal{S}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{F}\mathbf{f}\Big\}(\boldsymbol{\theta}) - \Big\{\mathcal{F}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{S}\mathbf{f}\Big\}(\boldsymbol{\theta}) = \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{S}_{\boldsymbol{\eta}}\mathbf{f},$$

$$v(\boldsymbol{\theta}) = \boldsymbol{\theta}\,\boldsymbol{.}\,\Big[\mathcal{S},\boldsymbol{\eta}\times,\mathcal{F}\Big]_{\boldsymbol{\theta}}\mathbf{f} = \boldsymbol{\theta}\,\boldsymbol{.}\,\Big\{\mathcal{S}\boldsymbol{\eta}\times\mathcal{F}\mathbf{f}\Big\}(\boldsymbol{\theta}) - \boldsymbol{\theta}\,\boldsymbol{.}\,\Big\{\mathcal{F}\boldsymbol{\eta}\times\mathcal{S}\mathbf{f}\Big\}(\boldsymbol{\theta}) = \boldsymbol{\theta}\,\boldsymbol{.}\,\mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \boldsymbol{\theta}\,\boldsymbol{.}\,\mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{S}_{\boldsymbol{\eta}}\mathbf{f},$$

where through $[\mathcal{A},\mathcal{B},\mathcal{C}]$ we denote the generalized commutator, $[\mathcal{A},\mathcal{B},\mathcal{C}] = \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{C}\mathcal{B}\mathcal{A}$.

Theorem 3. *For any functions $u, v \in H^1(\mathbb{S}^2)$ the following identities take place*

$$\nabla u(\boldsymbol{\theta}) = \underbrace{\frac{\nabla}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\{\mathcal{F}^{(\boldsymbol{\beta})}\nabla u\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{.}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{even\,part} + \underbrace{\frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\boldsymbol{\eta}\Delta\{\mathcal{F}u\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{.}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{odd\,part}, \tag{8}$$

$$\nabla^{\perp} v(\boldsymbol{\theta}) = \underbrace{-\frac{\nabla^{\perp}_{\boldsymbol{\theta}}}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\{\mathcal{F}^{(\boldsymbol{\tau})}\nabla^{\perp} v\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{.}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{odd\,part} + \underbrace{\frac{\boldsymbol{\theta}\times}{4\pi}\int\limits_{\mathbb{S}^2}\frac{\boldsymbol{\eta}\Delta\{\mathcal{F}v\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{.}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta}}_{even\,part}. \tag{9}$$

The analytic inversion formulas for operators $\mathcal{F}^{(\boldsymbol{\tau})}$ and $\mathcal{F}^{(\boldsymbol{\beta})}$ follow from the Theorem 3. Let $\mathbf{f}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} u(\boldsymbol{\theta}) + \nabla^{\perp}_{\boldsymbol{\theta}} v(\boldsymbol{\theta})$ is an odd vector field, $\mathbf{f}(-\boldsymbol{\eta}) = -\mathbf{f}(\boldsymbol{\eta})$. It is obvious that for even vector fields $\mathbf{f}(-\boldsymbol{\eta}) = \mathbf{f}(\boldsymbol{\eta})$ the $\mathcal{F}^{(\boldsymbol{\tau})}\mathbf{f}$ will be zero. We also know that $\mathcal{F}^{(\boldsymbol{\tau})}\nabla u = 0$, so the original vector field is not completely determined by its transformation $\mathcal{F}^{(\boldsymbol{\tau})}$. We see that the first term in the formula (9) gives the inversion formula. So we define only the stream function v_{odd} and, accordingly, only the solenoidal part $\nabla^{\perp} v_{odd}(\boldsymbol{\theta})$ of the vector field $\mathbf{f}$.

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ON REALIZABILITY OF GAUSS DIAGRAMS AND CONSTRUCTION OF MEANDERS

VIKTOR LOPATKIN (JOINT WORK WITH ANDREY GRINBLAT)

The problem of which Gauss diagram can be realized by knots is an old one [1] and has been solved in several ways [3],[4],[5]. However all these ways are indirect; they rest upon deep and nontrivial auxiliary construction. There is a natural question: whether one can arrive at these conditions in a more direct and natural fashion?

In this talk, we present a direct approach to this problem. We show that the needed conditions for realizability of a Gauss diagram can be interpreted as follows "the number of exits = the number of entrances" and the sufficient condition is based on Jordan curve Theorem.

We believe that the conditions for realizability of a Gauss diagram (by some plane curve) should be obtained in a natural manner; they should be deduced from an intrinsic structure of the curve.

In this talk, we suggest an approach, which satisfies the above principle. We use the fact that every Gauss diagram $\mathfrak{G}$ defines a (virtual) plane curve $\mathscr{C}(\mathfrak{G})$ (see [2, Theorem 1.A]), and the following simple ideas:

(1) For every chord of a Gauss diagram $\mathfrak{G}$, we can associate a closed path along the curve $\mathscr{C}(\mathfrak{G})$.
(2) For every two non-intersecting chords of a Gauss diagram $\mathfrak{G}$, we can associate two closed paths along the curve $\mathscr{C}(\mathfrak{G})$ such that every chord crosses both of those chords correspondences to the point of intersection of the paths.
(3) If a Gauss diagram $\mathfrak{G}$ is realizable (say by a plane curve $\mathscr{C}(\mathfrak{G})$), then for every closed path (say) $\mathscr{P}$ along $\mathscr{C}(\mathfrak{G})$ we can associate a coloring another part of $\mathscr{C}(\mathfrak{G})$ into two colors (roughly speaking we get "inner" and "outer" sides of $\mathscr{P}$ *cf.* Jordan curve Theorem). If a Gauss diagram is not realizable then ([2, Theorem 1.A]) it defines a virtual plane curve $\mathscr{C}(\mathfrak{G})$. We shall show that there exists a closed path along $\mathscr{C}(\mathfrak{G})$ for which we cannot associate a well-defined coloring of $\mathscr{C}(\mathfrak{G})$, *i.e.,* $\mathscr{C}(\mathfrak{G})$ contains a path is colored into two colors.

Using these ideas we solve the problem of which Gauss diagram can be realized by curves. We then give a matrix approach of realization of Gauss diagrams and then we present an algorithm to construct meanders

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FROM HARMONIC MAPPINGS TO RICCI SOLITONS BY INFINITESIMAL HARMONIC TRANSFORMATIONS

SERGEY STEPANOV

The purpose of the present report is the study of certain connections between the theory of *infinitesimal harmonic transformations* (see [1]) and the well known theory of *Ricci solitons* (see, for example, [2]). But we will begin our report with considering a new point of view on classical results of the global geometry of *harmonic mappings* (see [3]). The Bochner technique and its generalized version will help us to relate these various research topics (see [4]).

The report is organized as follows. In the first section of the report, we give brief survey of basic facts of the geometry "in the large" of harmonic mappings between Riemannian manifolds. We shall prove that the classical theorems on harmonic mappings are consequences of well-known assertions on subharmonic functions. Results of the second section of our paper with the title "Infinitesimal harmonic transformations" are obtained as analogs of results of the first section of the report. In turn, the results of the third section which has the title "Ricci solitons" are applications of the results of the second section of our report.

Theorems and corollaries of this report complement our results from [5] and [6].

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ON GEOMETRY OF GROMOV–HAUSDORFF SPACE

ALEXEY A. TUZHILIN

The Gromov–Hausdorff (GH-) distance between two metric spaces X and Y is a measure of difference between these spaces. To be more precise, let us isometrically embed X and Y into other metric spaces Z, in all possible ways, and calculate the least possible Hausdorff distance between the images. The resulting value is called the GH-distance between X and Y [1]. There are various beautiful applications of this notion like Gromov's theorem on groups of polynomial growth [2] and Gromov's compactness theorem [3, 4].

The most common use of the distance is related to description of the corresponding convergence. In the present talk we shall speak about another aspect. Namely, we discuss the geometry and topology of the first natural space endowed with the GH-distance, namely, the space $\mathcal{M}$ of isometry classes of compact metric spaces. It is well-know that GH-distance on $\mathcal{M}$ is a metric, and $\mathcal{M}$ with this metric is usually called the *Gromov–Hausdorff space*. We shall discuss the both classical and recent results devoted to $\mathcal{M}$, in particular, its local and global symmetries. For more details see [5]–[15].

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